ON THE NONEXISTENCE OF FREE DIFFERENTIABLE S^3 -ACTIONS ON HOMOTOPY SPHERES

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1. Introduction. The only known results concerning the nonexistence of free differentiable S^1 -actions on homotopy spheres are the following theorems of Montgomery, Yang and R. Lee:

THEOREM 1 (MONTGOMERY, YANG [1], [6]). *Let Σ_M^7 denote the Milnor 7-sphere. Then a homotopy 7-sphere Σ^7 admits a free S^1 -action if and only if $\Sigma^7 \approx k \Sigma_M^7$ for some $k \equiv 0, \pm 4, \pm 6, \pm 10$ or $14 \pmod{28}$.*

THEOREM 2 (R. LEE [1]). *There exist homotopy spheres in dimensions $8k+1$ ($k \geq 1$) which do not admit free S^1 -actions.*

In this note, we shall study the corresponding problem for free differentiable S^3 -actions. More precisely, we shall prove the following results:

THEOREM 3. *Let Σ_M^{11} denote the Milnor sphere which is the generator of θ_{11} . Then a homotopy 11-sphere $k \Sigma_M^{11}$ admits no differentiable free S^3 -actions if $k \not\equiv 0 \pmod{8}$.*

THEOREM 4. *If a homotopy 7-sphere Σ^7 admits a free differentiable S^3 -action, then Σ^7 is the standard sphere S^7 .*

It follows from Theorem 1 and Theorem 4 that there are free S^1 -actions on some homotopy 7-spheres which cannot be imbedded in a free S^3 -action. The proofs will be based on the computation of the Eells-Kuiper μ -invariants [2].

2. Proof of Theorem 3. It is well known that $\theta_{11} = Z_{992}$ [5], and the Eells-Kuiper invariant $\mu(\Sigma_M^{11}) \equiv -1/992 \pmod{1}$. Hence μ is a complete invariant for homotopy 11-spheres [2]. Theorem 3 is an immediate consequence of the following lemma:

LEMMA. *Let S^3 act freely and differentiably on a homotopy 11-sphere Σ^{11} . Then $\mu(\Sigma^{11}) \equiv n/124 \pmod{1}$ for some integer n .*

PROOF. Since S^3 acts freely on Σ^{11} , we have the principal bundle $\xi: S^3 \rightarrow \Sigma^{11} \rightarrow \Sigma^{11}/S^3$ which is homotopically equivalent to the Hopf

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bundle $S^3 \rightarrow S^{11} \rightarrow QP^2$, where QP^2 denotes the quaternionic projective plane. Hence the total Pontrjagin class $p(\xi)$ of the bundle ξ is given by $p(\xi) = 1 + 2\alpha + \alpha^2$, where α is the generator of $H^4(\sum^{11}/S^3; \mathbb{Z})$. Let W be the total space of the associated disk bundle $D^4 \rightarrow W \xrightarrow{\tau} \sum^{11}/S^3$ of ξ which is homotopically equivalent to $\sum^{11}/S^3$. Clearly W is a spin manifold with index $\tau(W) = 0$. Let $p_1(\sum^{11}/S^3) = r\alpha$, $r \in \mathbb{Z}$, and $\beta = \pi^*\alpha \in H^4(W; \mathbb{Z})$. By Hirzebruch's index theorem [3, p. 86], we have

$$p_2(\sum^{11}/S^3) = (45 + r^2)\alpha^2/7.$$

It is not difficult to see that $p_1(W) = (r+2)\beta$, and

$$p_2(W) = (r^2 + 14r + 52)\beta^2/7.$$

Thus the Eells-Kuiper μ -invariant has the following form:

$$\begin{aligned} \mu(\sum^{11}) &\equiv \{4p_2(W)p_1(W) - 3p_1^3(W) - 24r\}[W]/2^{11} \cdot 3 \cdot 31 \pmod{1} \\ &\equiv -(r+2)(r-2)(17r+62)/2^{11} \cdot 3 \cdot 7 \cdot 31 \pmod{1}. \end{aligned}$$

As μ is a complete invariant of homotopy 11-spheres, hence $992\mu(\sum^{11})$ is an integer modulo 992 with

$$992\mu(\sum^{11}) \equiv -(r+2)(r-2)(17r+62)/2^6 \cdot 3 \cdot 7 \pmod{992}.$$

We can see that the integer r must be in the form $r = 4m + 2$ for some integer m so that

$$992\mu(\sum^{11}) \equiv -m(m+1)(17m+24)/21 \pmod{992}.$$

This implies

$$(1) \quad m(m+1)(17m+24) \equiv 0 \pmod{21}.$$

The $\hat{A}_2$ -genus of the manifold $\sum^{11}/S^3$ is an integer because $\sum^{11}/S^3$ is a closed spin manifold [3, p. 198]. Thus we have $m(m+1) = 56i$ for some integer i because

$$\begin{aligned} \hat{A}_2(\sum^{11}/S^3) &= \{-4p_2(\sum^{11}/S^3) + 7p_1^2(\sum^{11}/S^3)\}/2^7 \cdot 45 \\ &= (r^2 - 4)/2^7 \cdot 7 = m(m+1)/56. \end{aligned}$$

Substituting this result in (1) we obtain $8i(17m+24) \equiv 0 \pmod{3}$, or equivalently $i(17m+24)/3$ is an integer. Therefore

$$\begin{aligned} \mu(\sum^{11}) &\equiv -m(m+1)(17m+24)/992 \cdot 21 \pmod{1} \\ &\equiv -8\{i(17m+24)/3\}/992 \pmod{1} \\ &\equiv -\{i(17m+24)/3\}/124 \pmod{1}. \end{aligned}$$

This completes the proof.

3. **Proof of Theorem 4.** The proof is exactly the same as the proof of Theorem 3. We have the principal bundle $\xi: S^3 \rightarrow \Sigma^7 \rightarrow \Sigma^7/S^3$ with associated disk bundle $D^4 \rightarrow W \rightarrow \Sigma^7/S^3$, and indexes $\tau(W)=1$, $\tau(\Sigma^7/S^3)=0$. Thus $p_1(\Sigma^7/S^3)=0$ by the Hirzebruch index theorem. We can easily see that $p_1(W)=2\beta$, where β is the generator of $H^4(W; \mathbb{Z})$, and

$$\begin{aligned}\mu(\Sigma^7) &\equiv \{p_1(W)^2 - 4\tau\} [W]/2^7 \cdot 7 \pmod{1} \\ &\equiv (4 - 4)/2^7 \cdot 7 \equiv 0 \pmod{1}.\end{aligned}$$

Hence $\Sigma^7 \approx S^7$ by [2].

In conclusion we conjecture that the homotopy sphere $k \Sigma_M^{4n-1}$, k odd, admits no free differentiable S^1 (resp. S^3)-actions if $n \geq 3$ (resp. $n \geq 4$), where Σ_M^{4n-1} denotes the Milnor sphere.

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