SHORTER NOTES

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EVERY OPERATOR IS THE SUM OF TWO IRREDUCIBLE ONES

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Let $\mathcal{H}$ be a separable (complex) Hilbert space. An operator on $\mathcal{H}$ is called *irreducible* if it has no reducing subspaces other than the trivial ones, $\{0\}$ and $\mathcal{H}$. Halmos [2] has recently aroused interest in these operators by showing that they are dense in the algebra of all operators. The present note was motivated by a paper of Fillmore and Topping [1] in which it is proved that every operator is the sum of four irreducible operators. We shall make use of the obvious fact that A is irreducible if and only if the only subspaces of $\mathcal{H}$ invariant under both Re A and Im A are $\{0\}$ and $\mathcal{H}$.

LEMMA. *Let $\mathcal{S}$ be a finite or countably infinite set of nonscalar operators on $\mathcal{H}$. Then there exists a hermitian operator K on $\mathcal{H}$ such that no member of $\mathcal{S}$ leaves invariant a nontrivial invariant subspace of K.*

The special case of this lemma, where $\mathcal{S}$ has one element, is proved in [3]. The Baire-Category proof given there immediately extends to the more general case.

THEOREM. *Every operator on a separable Hilbert space is the sum of two irreducible operators.*

PROOF. Let A be any operator on $\mathcal{H}$. If A is scalar, then any irreducible operator B will give the desired decomposition $A=(A-B)+B$. Hence assume A is not scalar and let $M=\operatorname{Re} A$ and $N=\operatorname{Im} A$.

Assume first that both M and N are nonscalar. Apply the lemma with $\mathcal{S}=\{M, N\}$ to obtain a hermitian operator K. Then $A=A_1+A_2$, where

$$A_1=(M-K)-iK \quad \text{and} \quad A_2=K+i(N+K).$$

Since every subspace invariant under $M-K$ and K is also invariant under M, the choice of K implies that A_1 is irreducible. So is A_2 by a similar argument.

Since M and N are not both scalar, to complete the proof we must only treat the case where exactly one of them is scalar. Assume, con-

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sidering iA instead of A if necessary, that M is scalar: $M=cI$. Apply the lemma with $\mathcal{S}=\{N\}$ to obtain K. Then

$$A_1 = K + cI + iN/2 \quad \text{and} \quad A_2 = -K + iN/2$$

are both irreducible and $A=A_1+A_2$.

References

1. Peter A. Fillmore and David M. Topping, *Sums of irreducible operators*, Proc. Amer. Math. Soc. **20** (1969), 131–133.

2. Paul R. Halmos, *Irreducible operators*, Michigan Math. J. **15** (1968), 215–223.

3. Heydar Radjavi and Peter Rosenthal, *Matrices for operators and generators of* $B(\mathcal{H})$, (to appear).

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