THE ARF INVARIANT FOR KNOT TYPES

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The purpose of this paper is to prove Theorem 1 below which gives a simple relation between the Arf invariant $\phi(k)$ and the Minkowski unit $C_p(k)$, $p=2$, of a tame knot k in 3-space.

THEOREM 1. $C_2(k) = (-1)^{\phi(k)}$.

The cobordism invariance of $\phi(k)$ which is proved by Robertello [3] follows from this theorem and Corollary 3.5 in [2]. Further, if we denote by $\Delta(t)$ the Alexander polynomial of k, then a simple calculation leads to the following

THEOREM 2. $\phi(k)=0$ *iff* $\Delta(-1) \equiv \pm 1 \pmod 8$.

1. Seifert matrix. Let k be an oriented tame knot in 3-space and let S be a Seifert surface of k. S is a 2-cell with $2h$ bands $B_1, \cdots, B_{2h}$, where h is the genus of S. Let $V_{2h}=(v_{ij})$ be the Seifert matrix associated to S. V_{2h} is an integral $2h\times 2h$ matrix and the (i, j) entry of the symmetric matrix $M=V_{2h}+V'_{2h}$ is odd iff $(i, j)=(2r-1, 2r)$ or $(2r, 2r-1)$, and hence $\det M$ is odd.

Let V_m $(m\leqq 2h)$ be the principal minor consisting of the first m rows and columns of V_{2h}. Then V_{2l} can be considered as the Seifert matrix associated to the surface S' obtained from S by removing $2h-2l$ bands $B_{2l+1}, \cdots, B_{2h}$. Let $D_m=\det(V_m+V'_m)$.

LEMMA 1. *Let* $1\leqq n\leqq h$ *and* $D_0=1$. *Then* $D_{2n-2}D_{2n}\equiv -1$ *or* $3 \pmod 8$ *according as* $v_{2n-1,2n-1}v_{2n,2n}$ *is even or odd. Moreover,* D_{2n-1} *is even, but if* $v_{2n-1,2n-1}$ *is odd then* $D_{2n-1}\equiv 2 \pmod 4$.

PROOF. Let $M=V_{2n}+V'_{2n}$. We know[1] that

$$D_{2n-2}D_{2n} = D_{2n-1}\det \tilde{M}(2n-1) - \left\{\det \tilde{M}\binom{2n-1}{2n}\right\}^2.$$

(For a proof, for example, see [1, p. 7].) Since

$$\det \tilde{M}\binom{2n-1}{2n}$$

is odd,

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[1] $\tilde{M}\binom{p}{q}$ denotes the matrix obtained from M by deleting the pth row and qth column, and $\tilde{M}(p)=\tilde{M}\binom{p}{p}$.

$$\left\{\det \tilde{M}\begin{pmatrix}2n-1\\2n\end{pmatrix}\right\}^2 \equiv 1 \pmod 8.$$

Hence, $D_{2n-2}D_{2n} \equiv D_{2n-1} \det \tilde{M}(2n-1)-1 \pmod 8$. Further, $\det \tilde{M}(2n-1) \equiv 2v_{2n,2n}D_{2n-2} \pmod 4$, $D_{2n-1} \equiv 2v_{2n-1,2n-1}D_{2n-2} \pmod 4$ and D_{2n-2} is odd. Therefore, $\det \tilde{M}(2n-1)\ D_{2n-1} \equiv 4v_{2n-1,2n-1}\ v_{2n,2n} \pmod 8$ and $D_{2n-1} \equiv 2v_{2n-1,2n-1} \pmod 4$. This proves Lemma 1.

Since $D_{2n-2}D_{2n} \equiv -1 \pmod 4$ by Lemma 1, it follows

LEMMA 2. $D_{2n} \equiv (-1)^n \pmod 4$ *for* $0 \leqq n \leqq h$.

Let $(a, b)_2$ denote the Hilbert symbol. Then Lemma 2 implies

LEMMA 3. $(-1, D_{2n})_2 = (-1)^n$.

Further, we can prove

LEMMA 4. $(D_{2n-1}, -D_{2n-2}D_{2n})_2 = 1$ *or* -1 *according as* $v_{2n-1,2n-1}v_{2n,2n}$ *is even or odd.*

PROOF. Let us write $D_{2n-1} = 2^m t$, where m is a nonnegative integer, t is odd and let $q = D_{2n-2}D_{2n}$. Then $(D_{2n-1}, -q)_2 = (2, -q)_2^m(t, -q)_2$. If $v_{2n-1,2n-1}v_{2n,2n}$ is even, then $q \equiv -1 \pmod 8$ by Lemma 1. Hence $(2, -q)_2 = 1$ and $(t, -q)_2 = 1$. Thus $(D_{2n-1}, -q)_2 = 1$. If $v_{2n-1,2n-1}v_{2n,2n}$ is odd, then D_{2n-1} is not divisible by 4, i.e. $m = 1$, and $q \equiv 3 \pmod 8$. Therefore, $(D_{2n-1}, -q)_2 = (2, -q)_2(t, -q)_2 = -1$.

2. **Proof of Theorem 1.** Given a Seifert matrix V_{2h} of a knot k, we can define the Arf invariant $\phi(k)$ [3] and the Minkowski unit $C_2(k)$ [2] as follows.

$$\phi(k) \equiv \sum_{i=1}^{h} v_{2i-1,2i-1}v_{2i,2i} \pmod 2, \tag{2.1}$$

and

$$C_2(k) = (-1)^\beta(-1, -D_{2h})_2 \prod_{i=1}^{2h-1}(D_i, -D_{i+1})_2, \tag{2.2}$$

where $\beta = [h/2] + (1+h)(D_{2h}+1)/2$.

Since $D_{2h} \equiv (-1)^h \pmod 4$ by Lemma 2, we see that $\beta \equiv [h/2] + h + 1 \pmod 2$.

Now Lemma 2 shows that D_{2n} is not zero, while D_{2n-1} may be zero. If D_{2n-1} is zero, $(D_{2n-2}, -D_{2n-1})_2$ and $(D_{2n-1}, -D_{2n})_2$ are interpreted to be $(D_{2n-2}, -1)_2$ and $(1, -D_{2n})_2$, respectively.

Now the proof of Theorem 1 will proceed by induction on h, the genus of S.

For $h=0$, the theorem is obvious. Suppose $h=1$. Since $\beta=1$ and $(-1,-D_2)_2=1$ by Lemma 3, it suffices to show that

$$(D_1, -D_2)_2 = (-1)^{v_{11}v_{22}}. \tag{2.3}$$

If $D_1\neq 0$, then (2.3) follows from Lemma 4. If $D_1=0$, that is, $v_{11}=0$, we have to show that $C_2(k)=1$. However, since $C_2(k)$ depends only on the R-equivalent class [2], to calculate $C_2(k)$ we may use $RMR'=(a_{ij})$ instead of $M=V_2+V_2'$ for some integral unimodular matrix R. R may be chosen so that $a_{11}\neq 0$ and $a_{22}=0$. Then $C_2(k)=(a_{11}, a_{12}^2)_2=1$.

Now we suppose that the theorem is true for $n<h$, and proceed to prove it for $n=h$.

Consider the surface S' obtained from S by removing the last two bands B_{2h-1} and B_{2h}. The genus of S' is $h-1$ and the boundary of S' is a new knot k'. Moreover, S' is a Seifert surface of k' and V_{2h-2} is the Seifert matrix associated to S'. Thus, by the induction assumption, we see that in order to prove the theorem it is sufficient to show that

$$\begin{aligned}(-1)^{\beta+\beta'}(-1, -D_{2h})_2(D_{2h-2}, -D_{2h-1})_2(D_{2h-1}, -D_{2h})_2(-1, -D_{2h-2})_2\\ = (-1)^{v_{2h-1,2h-1}v_{2h,2h}},\end{aligned} \tag{2.4}$$

where $\beta'=[(h-1)/2]+h$.

Since $\beta+\beta'\equiv h \pmod 2$ and

$$(-1, -D_{2h})_2(-1, -D_{2h-2})_2 = (-1)^{h+1}(-1)^h = -1$$

by Lemma 3, (2.4) reduces to

$$(-1)^{h+1}(D_{2h-2}, -D_{2h-1})_2(D_{2h-1}, -D_{2h})_2 = (-1)^{v_{2h-1,2h-1}v_{2h,2h}}. \tag{2.5}$$

If $D_{2h-1}=0$, then $v_{2h-1,2h-1}v_{2h,2h}$ must, by Lemma 1, be an even number. The left-hand side of (2.5) is, by our convention,

$$(-1)^{h+1}(D_{2h-2}, -1)_2(1, -D_{2h})_2 = (-1)^{h+1}(-1)^{h-1} = 1.$$

Suppose $D_{2h-1}\neq 0$. Then, since

$$\begin{aligned}(D_{2h-2}, -D_{2h-1})_2 &= (D_{2h-2}, -1)_2(D_{2h-2}, D_{2h-1})_2\\ &= (-1)^{h-1}(D_{2h-2}, D_{2h-1})_2,\end{aligned}$$

(2.5) reduces to

$$(D_{2h-1}, -D_{2h-2}D_{2h})_2 = (-1)^{v_{2h-1,2h-1}v_{2h,2h}}. \tag{2.6}$$ =ing

This follows from Lemma 4, however. Thus the proof of Theorem 1 is complete.

Since Theorem 2 can be proved easily by induction on h, the details will be omitted.

REFERENCES

1. B. W. Jones, *The arithmetic theory of quadratic forms*, The Carus Math. Monographs No. 10, Math. Assoc. of America, Buffalo, N. Y., 1950.

2. K. Murasugi, *On the Minkowski unit of slice links*, Trans. Amer. Math. Soc. **114** (1965), 377–383.

3. R. A. Robertello, *An invariant of knot cobordism*, Comm. Pure Appl. Math. **18** (1965), 543–555.

4. H. Seifert, *Über das Geschlecht von Knoten*, Math. Ann. **110** (1934), 571–592.

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