THE RADON-NIKODYM THEOREM AND THE MEAN CONVERGENCE OF BANACH SPACE VALUED MARTINGALES

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In 1943, Phillips [3] extended the classical Radon-Nikodym Theorem to the setting of Banach space valued set functions. Later, in the early sixties, Chatterji [1] and Scalora [4] proved a theorem dealing with the norm convergence of martingales in the L^p ($1 \leq p < \infty$) of Banach space valued functions. Even though there has been much discussion dealing with the relationships between martingale convergence and derivations, there seems to be no explicit statement of the fact that the Chatterji-Scalora mean martingale convergence theorem can be deduced directly from Phillips' Theorem. Equally, there appears to be no mention that a strengthened version of the Chatterji-Scalora theorem directly implies (an important special case of) Phillips' Radon-Nikodym Theorem. The purpose of this note is to demonstrate these facts.

Throughout this note, (Ω, Σ, μ) is a fixed finite measure space; $\mathfrak{X}$ is a Banach space, and $L^p(\Omega, \Sigma, \mu, \mathfrak{X})$ ($=L^p(\mathfrak{X})$), $1 \leq p < \infty$, is the Banach space of all measurable $\mathfrak{X}$ -valued (equivalence classes of) functions f defined on Ω which satisfy $\|f\|_p = (\int_{\Omega} \|f\|^p d\mu)^{1/p} < \infty$. By a partition $\pi = \{E_n\}_{n=1}^m$ of Ω , we shall mean a finite set of disjoint sets in Σ such that $\bigcup_{n=1}^m E_n = \Omega$.

DEFINITION. An additive set function F defined on Σ and having its values in $\mathfrak{X}$ is said to be of p -bounded variation ($1 \leq p < \infty$) if

$$\|F\|_p = \left(\sup_{\pi} \sum_{\pi} \left(\frac{\|F(E_n)\|}{\mu(E_n)} \right)^p \mu(E_n) \right)^{1/p}$$

where the supremum is taken over all partitions $\pi = \{E_n\}$ and the convention $0/0 = 0$ is observed here and throughout.

The following is a special, but important, case of Phillips' generalization of the Radon-Nikodym Theorem.

THEOREM I [3]. *If $\mathfrak{X}$ is reflexive and $F: \Sigma \rightarrow \mathfrak{X}$ is countably additive, μ -continuous, and of p -bounded variation ($1 \leq p < \infty$), then there exists $f \in L^p(\mathfrak{X})$ such that $F(E) = \int_E f d\mu$ for all $E \in \Sigma$, where the integral is the Bochner integral.*

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Leaving, for the time being, the Radon-Nikodym-Phillips Theorem, we shall now establish the setting for a generalization of the Chatterji-Scalora mean martingale convergence theorem.

DEFINITION. Let T be a directed set and $\{B_t, t \in T\}$ be an increasing net of sub- σ -fields of Σ (i.e. if $t_1, t_2 \in T$ and $t_1 \leq t_2$, then $B_{t_1} \subset B_{t_2}$). $\{f_t, B_t, t \in T\}$ is an $\mathfrak{X}$ -valued martingale in $L^p(\mathfrak{X})$ if $f_t \in L^p(\mathfrak{X})$ for all $t \in T$, and if $t, t_0 \in T$ and $t \geq t_0$, then $E^{B_{t_0}}(f_t) = f_{t_0}$, where $E^{B_{t_0}}$ is the conditional expectation operator relative to B_{t_0} .

Here we understand that if B is a sub- σ -field of Σ and $f \in L^p(\mathfrak{X})$, $E^B(f)$ is the unique member of $L^p(\mathfrak{X})$ which is B -measurable and satisfies

$$\int_E E^B(f) d\mu = \int_E f d\mu \quad \text{for all } E \in B.$$

The following theorem, in the case that the directed set T is the positive integers with the natural order, is due to Chatterji [1] and Scalora [4]. It is the aim of this note to demonstrate the equivalence of the following theorem and Theorem I.

THEOREM II. Let $\mathfrak{X}$ be reflexive and let $\{f_t, B_t, t \in T\}$ be an $\mathfrak{X}$ -valued martingale in $L^p(\mathfrak{X})$.

(a) If $1 < p < \infty$ and there exists M such that $\|f_t\|_p \leq M$ for all $t \in T$, then the net $\{f_t, t \in T\}$ converges in the $L^p(\mathfrak{X})$ norm.

(b) In the case $p = 1$, if there exists a constant M such that $\|f_t\|_1 \leq M$ for all $t \in T$ and a parameter $t_0 \in T$ such that for any $\epsilon > 0$ there corresponds a $\delta > 0$ such that $t \geq t_0$ and $\mu(E) < \delta$ implies $\int_E \|f_t\| d\mu < \epsilon$, then the net $\{f_t, t \in T\}$ converges in the $L^1(\mathfrak{X})$ norm.

We shall show that Theorems I and II are equivalent.

I $\rightarrow$ II. Assume first that $1 < p < \infty$. By the Hölder inequality and the fact that $\|f_t\|_p \leq M$ for $t \in T$, we have, for $p^{-1} + q^{-1} = 1$, and $E \in \Sigma$,

$$\int_E \|f_t\| d\mu \leq \|f_t\|_p \|\chi_E\|_q \leq M \|\chi_E\|_q.$$

(χ_E is the indicator function of E .) Since for $1 \leq q < \infty$, $\lim_{\mu(E) \rightarrow 0} \|\chi_E\|_q = 0$, it follows that for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E \|f_t\| d\mu$ for all $t \in T$.

Now let $1 \leq p < \infty$. Since $\{B_t, t \in T\}$ is an increasing net of sub- σ -fields of Σ , it follows easily that $\bigcup_{t \in T} B_t$ is a subfield of Σ and hence generates a sub- σ -field, Σ_0 , of Σ . Now we shall define an $\mathfrak{X}$ valued set function F on Σ_0 by

$$F(E) = \lim_t \int_E f_t d\mu, \quad E \in \Sigma_0.$$

Showing that such a set function exists requires some discussion. To show that $\lim_t \int_E f_t d\mu$ exists strongly in $\mathfrak{X}$ for each $E \in \Sigma_0$, choose $t_0 \in T$ such that given any $\epsilon > 0$ there exists a $\delta (= \delta(\epsilon)) > 0$ such that $t \geq t_0$ implies $\int_E \|f_t\| d\mu < \epsilon$ whenever $\mu(E) > \delta$. In the case $p = 1$, the existence of such a t_0 is guaranteed by hypothesis. In the case $1 < p < \infty$, as the beginning of the proof shows, any $t \in T$ will qualify as the t_0 above. Now, let $\epsilon > 0$ be fixed, but arbitrary, and choose $\delta > 0$ such that $\int_E \|f_t\| d\mu < \epsilon/4$ for $t \geq t_0$ and $\mu(E) < \delta$. If $E \in \Sigma_0$, according to [2, III. 7.1], there exists a set $S \in \bigcup_{t \in T} B_t$ such that both $\mu(E - S)$ and $\mu(S - E) < \delta$. Since $S \in \bigcup_{t \in T} B_t$ and $\{B_t, t \in T\}$ is an increasing net, there exists $t' \geq t_0$ such that $S \in B_{t'}$ for all $t \geq t'$. Moreover, for $t_1, t_2 \geq t'$,

$$(*) \quad \int_S f_{t_1} d\mu = \int_S E^{B_{t'}}(f_{t_1}) d\mu = \int_S f_{t_2} d\mu, \quad i = 1, 2,$$

since $\{f_t, B_t, t \in T\}$ is a martingale and $S \in B_{t'}$. Thus for $t_1, t_2 \geq t'$ the following inequalities hold:

$$\begin{aligned} \left\| \int_E f_{t_1} d\mu - \int_E f_{t_2} d\mu \right\| &\leq \left\| \int_E f_{t_1} d\mu - \int_S f_{t_1} d\mu \right\| \\ &\quad + \left\| \int_S f_{t_1} d\mu - \int_S f_{t_2} d\mu \right\| \\ &\quad + \left\| \int_S f_{t_2} d\mu - \int_E f_{t_2} d\mu \right\| \\ &\leq \int_{E-S} \|f_{t_1}\| d\mu + \int_{S-E} \|f_{t_1}\| d\mu \\ &\quad + \int_{E-S} \|f_{t_2}\| d\mu + \int_{S-E} \|f_{t_2}\| d\mu, \end{aligned}$$

by the triangle inequality and (*) above,

$$< 4(\epsilon/4) = \epsilon, \quad \text{by the choice of } t_0 \text{ and } S.$$

It follows that the set function F as defined above is well defined on Σ_0 . Moreover, according to [2, IV.10.6], F is countably additive on Σ_0 . In addition, F clearly vanishes on Σ_0 -sets of zero μ -measure. Thus, by a result of Pettis, [2, IV.10.1], F is μ -continuous on Σ_0 .

Next we shall show that F is of p -bounded variation.

$$\begin{aligned}
\|F\|_p^p &= \sup_{\pi} \sum \|F(E_n)\|_{\mu(E_n)}^{1-p} \\
&= \sup_{\pi} \lim_t \sum_{\pi} \left\| \int_{E_n} f_t d\mu \right\|_{\mu(E_n)}^{1-p}, \text{ by the definition of } F, \\
&\leq \sup_{\pi} \lim_t \sum_{\pi} \|f_t \chi_{E_n}\|_p^p \|\chi_{E_n}\|_q^p \mu(E_n)^{1-p}, \\
&\quad \text{by the Hölder inequality with } p^{-1} + q^{-1} = 1, \\
&= \sup_{\pi} \lim_t \sum_{\pi} \|f_t \chi_{E_n}\|_p^{p/q+1-p} \mu(E_n)^{p/q+1-p}, \text{ where if } q = \infty, p/q = 0, \\
&= \sup_{\pi} \lim_t \sum_{\pi} \int_{E_n} \|f_t\|^p d\mu, \text{ since } p/q + 1 - p = 0, \\
&= \sup_{\pi} \lim_t \int_{\Omega} \|f_t\|^p d\mu \leq M^p.
\end{aligned}$$

Therefore the hypothesis of Theorem I is satisfied by F . By Theorem I, there exists a Σ_0 -measurable $f \in L^p(\mathfrak{X})$ such that $F(E) = \int_E f d\mu$ for all $E \in \Sigma_0$.

To finish the proof, we shall show that $\lim_t f_t = f$ in the $L^p(\mathfrak{X})$ norm. First note that if $t_0 \in T$ is arbitrary and $t \geq t_0$, by the fact that $\{f_t, B_t, t \in T\}$ is a martingale,

$$\int_E f_t d\mu = \int_E E^{B_{t_0}}(f_t) d\mu = \int_E f_{t_0} d\mu$$

for all $E \in B_{t_0}$. From the definition of F and f , it follows immediately that $\int_E f d\mu = \int_E f_{t_0} d\mu$ for all $E \in B_{t_0}$. Accordingly, since f_{t_0} is B_{t_0} -measurable, $E^{B_{t_0}}(f) = f_{t_0}$. Moreover, if $\epsilon > 0$ is given, the Σ_0 -measurability guarantees, by [2, III.8.3], the existence of a simple function $g = \sum_{i=1}^m x_i \chi_{E_i}$, $x_i \in \mathfrak{X}$, $E_i \in \bigcup_{t \in T} B_t$ such that $\|f - g\|_p < \epsilon/2$. Now since $\{B_t, t \in T\}$ is an increasing net of sub- σ -fields of Σ , there exists a $\bar{t} \in T$ such that $E_i \in B_t$ for all $t \geq \bar{t}$ and all $i = 1, \dots, m$. Therefore $E^{B_t}(g) = g$ for all $t \geq \bar{t}$. Hence for $t \geq \bar{t}$, we have

$$\begin{aligned}
\|f - f_t\|_p &\leq \|f - g\|_p + \|g - f_t\|_p \\
&= \|f - g\|_p + \|E^{B_t}(g - f)\|_p, \\
&\quad \text{since } E^{B_t}(f) = f_t \text{ and } E^{B_t}(g) = g, \text{ for all } t \geq t_0, \\
&\leq 2\|f - g\|_p, \text{ since } E^{B_t} \text{ is a contraction,} \\
&\leq 2(\epsilon/2) = \epsilon, \text{ by the choice of } g.
\end{aligned}$$

This proves I $\rightarrow$ II.

II→I. Assume first that $1 < p < \infty$ and let F satisfy the hypothesis of Theorem I. For each partition $\pi = \{E_n\}$, form the simple function

$$f_\pi = \sum_{\pi} \frac{F(E_n)}{\mu(E_n)} \chi_{E_n}.$$

If $\sigma(\pi)$ denotes the (trivial) σ -field generated by $\pi = \{E_n\}$, and the collection P of all partitions is directed by the partial ordering of refinement, then $\{f_\pi, \sigma(\pi), \pi \in P\}$ becomes a martingale. Moreover,

$$\begin{aligned} \|f_\pi\|_p &= \left(\int_{\Omega} \left\| \sum_{\pi} \frac{F(E_n)}{\mu(E_n)} \chi_{E_n} \right\|^p d\mu \right)^{1/p} \\ &= \left(\sum_{\pi} \left\| \frac{F(E_n)}{\mu(E_n)} \right\|^p \mu(E_n) \right)^{1/p} \leq \|F\|_p. \end{aligned}$$

Thus the hypothesis of Theorem II is satisfied. By Theorem II, there exists $f \in L^p(\mathfrak{X})$ such that $\lim_{\pi} \|f_\pi - f\|_p = 0$. Now we shall show that f is the Radon-Nikodym-Phillips derivative of F . If $E \in \Sigma$,

$$\begin{aligned} \left\| F(E) - \int_E f d\mu \right\| &\leq \limsup_{\pi} \left\| F(E) - \int_E f_\pi d\mu \right\| \\ &\quad + \limsup_{\pi} \left\| \int_E f_\pi d\mu - \int_E f d\mu \right\|. \end{aligned}$$

The first term on the right is zero since if $E \in \pi_0$ and $\pi \geq \pi_0$, $\int_E f_\pi d\mu = F(E)$. On the other hand, by the Hölder inequality,

$$\begin{aligned} \limsup_{\pi} \left\| \int_E (f_\pi - f) d\mu \right\| &\leq \limsup_{\pi} \int_E \|f_\pi - f\| d\mu \\ &\leq \limsup_{\pi} \|f_\pi - f\|_p \|\chi_E\|_q \quad (p^{-1} + q^{-1} = 1) = 0. \end{aligned}$$

by Theorem II. Thus $F(E) = \int_E f d\mu$ for all $E \in \Sigma$. This proves II→I in the case $1 < p < \infty$.

The $L^1(\mathfrak{X})$ case will be proved by use of the $L^2(\mathfrak{X})$ case. Let $F: \Sigma \rightarrow \mathfrak{X}$ be μ -continuous, countably additive, and of bounded variation. By [3, Lemma 5.4], there exists a sequence $\{S_n\} \subset \Sigma$ such that $S_n \supset S_{n+1}$ and $\lim_{n \rightarrow \infty} \mu(S_n) = 0$ and such that $\|F(E)\| \leq n\mu(E)$ for all $E \in \Omega - S_n$. If F_n is defined by $F_n(E) = F(E - S_n)$, then clearly $\|F_n\|_2 < \infty$ for all n . Now if f_n is the $L^2(\mathfrak{X})$ -Radon-Nikodym-Phillips derivative of F_n , and we define f on Ω by $f(s) = f_n(s)$ for $s \in S_{n-1} - S_n$ where $S_0 = \Omega$, then f is measurable. Moreover, since F is μ -continuous and $\lim_{n \rightarrow \infty} \mu(S_n) = 0$,

it follows that $F(E) = \lim_{n \rightarrow \infty} F(E - S_n) = \lim_{n \rightarrow \infty} F_n(E) = \lim_{n \rightarrow \infty} \int_E f_n d\mu = \int_E f d\mu$. Also, since F is of bounded variation, we have $\int_{\Omega} \|f\| d\mu = \|F\|_1 < \infty$, and $f \in L^1(\mathfrak{X})$. Q.E.D.

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