SUMMABILITY OF A SEQUENCE OF FOURIER COEFFICIENTS BY A TRIANGULAR MATRIX TRANSFORMATION

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1. Definitions and notations. Let $\sum_{n=0}^{\infty} v_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. We shall consider a sequence to sequence transformation of the type

$$(1.1) \quad u_n = \sum_{k=0}^{\infty} d_{nk} s_k, \quad d_{nk} = 0 \quad \text{for } k > n,$$

in which the elements of the matrix $D = ((d_{nk}))$ are real or complex constants. The sequence $\{u_n\}$ is called the sequence of D -means of $\{s_n\}$. If $\lim_{n \rightarrow \infty} u_n$ exists and is equal to u then we say that the series $\sum v_n$ or the sequence $\{s_n\}$ is summable D to the sum u .

Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write $P_n = p_0 + p_1 + \dots + p_n \neq 0$, $P_{-1} = p_{-1} = 0$. Then the matrix D defines a Nörlund matrix (N, p_n) [6] if

$$(1.2) \quad d_{nk} = p_{n-k}/P_n, \quad (n \geq k \geq 0).$$

The conditions for the regularity of the (N, p_n) mean are

$$(1.3) \quad \lim_{n \rightarrow \infty} p_n/P_n = 0 \quad \text{and} \quad \sum_{k=0}^n |p_k| = O(|P_n|), \quad n \rightarrow \infty.$$

In the special case in which

$$(1.4) \quad p_n = \binom{n + \alpha - 1}{\alpha - 1} = \frac{\Gamma(n + \alpha)}{\Gamma(n + 1)\Gamma(\alpha)} \quad (\alpha > -1),$$

the (N, p_n) mean reduces to the familiar (C, α) mean. If $p_n = (n + 1)^{-1}$ the (N, p_n) mean is called the harmonic mean and is denoted by $(N, 1/(n + 1))$.

The product of the $(C, 1)$ matrix with a (N, p_n) matrix defines the matrix $(C, 1) \cdot (N, p_n)$. Thus the $(C, 1) \cdot (N, p_n)$ matrix is given by

$$(1.5) \quad d_{nk} = \frac{1}{n + 1} \sum_{r=k}^n p_{r-k}/P_r \quad (0 \leq k \leq n),$$

and the corresponding mean of $\{s_n\}$ is

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$$(1.6) \quad u_n = \sum_{k=0}^n \frac{1}{n+1} \sum_{r=k}^n p_{r-k} s_k / P_r = \frac{1}{n+1} \sum_{r=0}^n \frac{1}{P_r} \sum_{k=0}^r p_{r-k} s_k.$$

Similarly, one defines the $(N, p_n) \cdot (C, 1)$ summability.

Silverman has shown in [7] that a necessary and sufficient condition for a (N, p_n) matrix to be permutable with the $(C, 1)$ matrix is that it be a Cesàro matrix. This implies *inter alia* that $(C, 1) \cdot (N, p_n) \neq (N, p_n) \cdot (C, 1)$ except when $\{p_n\}$ is defined by (1.4).

Let $f(t)$ be integrable L in $(-\pi, \pi)$ and periodic with period 2π , and let

$$(1.7) \quad f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Then the conjugate series of (1.7) at $t=x$ is

$$(1.8) \quad \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x).$$

We write throughout:

$$\psi(t) = f(x+t) - f(x-t) - l, \text{ where } l \text{ is a finite number.}$$

$$\Psi(t) = \int_0^t |\psi(u)| du.$$

$$S_n = \frac{1}{P_n} \sum_{\nu=0}^n \frac{P_\nu}{\nu+1}; \quad \Delta_n \lambda_n = \lambda_n - \lambda_{n+1}.$$

$\tau = [1/t]$, $P_\tau = P(1/t)$, where τ denotes the greatest integer not greater than $1/t$.

K denotes a positive constant not necessarily the same at each occurrence.

2. Introduction. Concerning the $(C, 1)$ summability and the $(N, 1/(n+1)) \cdot (C, 1)$ summability of the sequence $\{nB_n(x)\}$, Mohanty and Nanda [5] and Varshney [8] have proved the following results.

THEOREM A. If $\psi(t) = o(\{\log(1/t)\}^{-1})$ as $t \rightarrow 0$ and

$$(2.1) \quad a_n = O(n^{-\delta}), \quad b_n = O(n^{-\delta}), \quad 0 < \delta < 1,$$

then the sequence $\{nB_n(x)\}$ is summable $(C, 1)$ to the value l/π .

THEOREM B. If

$$(2.2) \quad \Psi(t) = o \left\{ t \left(\log \frac{1}{t} \right)^{-1} \right\}, \quad t \rightarrow 0,$$

then the sequence $\{nB_n(x)\}$ is summable $(N, 1/(n+1)) \cdot (C, 1)$ to l/π .

It is apparent from Silverman's result [7] that the summability $(C, 1) \cdot (N, 1/(n+1))$ is different from the summability $(N, 1/(n+1)) \cdot (C, 1)$. However, it may be observed that the summability $(N, 1/(n+1))$ implies the (C, δ) summability for any $\delta > 0$ and, therefore, from a well-known result concerning the (C) summability, it follows that both $(C, 1) \cdot (N, p_n)$ as well as $(N, 1/(n+1)) \cdot (C, 1)$ imply the $(C, 1+\delta)$ summability, for some suitable $\{p_n\}$. In view of this it is natural to expect that the hypothesis (2.2) of Theorem B may lead to the summability $(C, 1) \cdot (N, 1/(n+1))$ of the sequence $\{nB_n(x)\}$. That this is indeed true is demonstrated by the special case $p_n = (n+1)^{-1}$ of our Theorem 1 stated in the next section.

It is known that if $\psi(t) = o(1)$, $t \rightarrow 0$, then the sequence $\{nB_n(x)\}$ is summable (C, r) , $r > 1$, to the value l/π and that it is a best possible result in the sense that it breaks down for $r = 1$ [9, p. 62]. This result follows as a corollary from our Theorem 2 when one observes that the sequence $\{p_n\}$ defined by (1.4) satisfies the hypotheses of Theorem 2 if $0 < \alpha < 1$.

3. We prove the following.

THEOREM 1. *Let $\{p_n\}$ be a nonnegative, monotonic nonincreasing sequence such that $\log n = O(P_n)$, $n \rightarrow \infty$. Then, if*

$$(3.1) \quad \Psi(t) = o(t/P_r), \quad t \rightarrow 0,$$

the sequence $\{nB_n(x)\}$ is summable $(C, 1) \cdot (N, p_n)$ to the value l/π .

THEOREM 2. *If $\psi(t) = o(1)$, $t \rightarrow 0$, and $\{p_n\}$ is a nonnegative, monotonic nonincreasing sequence such that $S_n = O(1)$, then the sequence $\{nB_n(x)\}$ is summable $(C, 1) \cdot (N, p_n)$ to the value l/π .*

4. We need the following lemmas for the proof of our theorems.

LEMMA 1. *If $\{p_n\}$ is a positive sequence such that $(n+1)p_n \leq KP_n$ and $S_n = O(1)$, then $P_n \rightarrow \infty$ as $n \rightarrow \infty$.*

PROOF. As pointed out by the present author in [1], the hypotheses of Lemma 1 imply that

$$\sum_{k=n+1}^{\infty} \frac{1}{(k+1)P_{k-1}} = O\left(\frac{1}{P_n}\right) \quad (n = 1, 2, 3, \dots),$$

and therefore $P_n \rightarrow \infty$ as $n \rightarrow \infty$. It may be remarked that with a slight modification in the reasoning given by the present author in [1] it is possible to avoid the hypothesis $(n+1)p_n \leq KP_n$.

LEMMA 2. *If $\{p_n\}$ is nonnegative and nonincreasing, then for $0 \leq a \leq b \leq \infty$ and $0 \leq t \leq \pi$,*

$$\left| \sum_{k=a}^b p_k \exp ikt \right| \leq KP_r.$$

Lemma 2 may be proved by following the technique of proof of Lemma 5.11 in McFadden [4].

LEMMA 3. *If $\{p_n\}$ is nonnegative and nonincreasing, then as $n \rightarrow \infty$*

$$\begin{aligned} \sum_{v=0}^n \frac{1}{P_v} \sum_{k=0}^v (v-k)p_k \exp i(v-k)t \\ = O(t^{-2}) + O\left(t^{-1}P_r \sum_{v=r}^n \frac{1}{P_v}\right) + O\left(\frac{nt^{-1}P_r}{P_{n+1}}\right) \end{aligned}$$

uniformly in $0 < t \leq \pi$.

PROOF. Applying Abel's transformation we get

$$\begin{aligned} \sum_{v=k}^n (v-k) \frac{1}{P_v} \exp i(v-k)t \\ = \sum_{v=k}^n \Delta_v \left(\frac{v-k}{P_v} \right) \sum_{\mu=k}^v \exp i(\mu-k)t + \frac{n-k+1}{P_{n+1}} \sum_{\mu=k}^n \exp i(\mu-k)t \\ = (1 - \exp it)^{-1} \left[\sum_{v=k}^n \Delta_v \left(\frac{v-k}{P_v} \right) \{1 - \exp i(v-k+1)t\} \right. \\ \left. + \frac{n-k+1}{P_{n+1}} \{1 - \exp i(n-k+1)t\} \right] \\ = (1 - \exp it)^{-1} \left[- \sum_{v=k}^n \frac{p_{v+1}}{P_v P_{v+1}} (v-k) \exp i(v-k+1)t \right. \\ \left. + \sum_{v=k}^n \frac{1}{P_{v+1}} \exp i(v-k+1)t \right. \\ \left. - \frac{n-k+1}{P_{n+1}} \exp i(n-k+1)t \right]. \end{aligned}$$

Changing the order of summation of the inner sums, thus we have

$$\begin{aligned}
\Sigma &\equiv \left| \sum_{\nu=0}^n \frac{1}{P_{\nu}} \sum_{k=0}^{\nu} (\nu - k) p_k \exp i(\nu - k)t \right| \\
&= \left| \sum_{k=0}^n p_k \sum_{\nu=k}^n \frac{(\nu - k)}{P_{\nu}} \exp i(\nu - k)t \right| \\
&\leq Kt^{-1} \left[\left| \sum_{k=0}^n p_k \sum_{\nu=k}^n \frac{p_{\nu+1}}{P_{\nu}P_{\nu+1}} (\nu - k) \exp i(\nu - k + 1)t \right| \right. \\
&\quad \left. + \left| \sum_{k=0}^n p_k \sum_{\nu=k}^n \frac{1}{P_{\nu+1}} \exp i(\nu - k + 1)t \right| \right. \\
&\quad \left. + \frac{1}{P_{n+1}} \left| \sum_{k=0}^n p_k (n - k + 1) \exp i(n - k + 1)t \right| \right] \\
&= \sum_1 + \sum_2 + \sum_3, \text{ say.}
\end{aligned}$$

Again, by a change of order of summation, we have

$$\begin{aligned}
\sum_1 &\leq Kt^{-1} \sum_{\nu=0}^n \frac{p_{\nu+1}}{P_{\nu}P_{\nu+1}} \left| \sum_{k=0}^{\nu} p_k (\nu - k) \exp i(\nu - k + 1)t \right| \\
&\leq Kt^{-1} \sum_{\nu=0}^{\tau-1} \frac{\nu p_{\nu+1}}{P_{\nu}P_{\nu+1}} \sum_{k=0}^{\nu} p_k \\
&\quad + Kt^{-1} \sum_{\nu=\tau}^n \frac{\nu p_{\nu+1}}{P_{\nu}P_{\nu+1}} \max_{0 \leq \rho \leq \nu} \left| \sum_{k=0}^{\rho} p_k \exp i(\nu - k + 1)t \right| \\
&\quad \text{(by Abel's Lemma, the first sum is taken as zero if } \tau=0) \\
&\leq Kt^{-2} + Kt^{-1}P_{\tau} \sum_{\nu=\tau}^n \frac{1}{P_{\nu}},
\end{aligned}$$

by virtue of Lemma 2 and the fact that $(\nu+1)p_{\nu} \leq P_{\nu}$.

Similarly,

$$\begin{aligned}
\sum_2 &\leq Kt^{-1} \sum_{\nu=0}^n \frac{1}{P_{\nu+1}} \left| \sum_{k=0}^{\nu} p_k \exp i(\nu - k + 1)t \right| \\
&\leq Kt^{-2} + Kt^{-1}P_{\tau} \sum_{\nu=\tau}^n \frac{1}{P_{\nu}},
\end{aligned}$$

by Lemma 2.

Finally, by Lemma 2 and Abel's Lemma, we have

$$\sum_3 \leq Kt^{-1}n P_{\tau} \frac{1}{P_{n+1}},$$

which completes the proof of the lemma.

5. **Proof of Theorem 1.** We have

$$\begin{aligned} nB_n(x) &= \frac{1}{\pi} \int_0^\pi \{f(x+t) - f(x-t)\} n \sin ntdt \\ &= \frac{1}{\pi} \int_0^\pi \psi(t) n \sin ntdt + \frac{l}{\pi} \{1 - (-1)^n\}. \end{aligned}$$

Therefore, if $\{t_n(x)\}$ denotes the (N, p_n) mean of $\{nB_n(x)\}$, then

$$\begin{aligned} t_n(x) - \frac{l}{\pi} &= \frac{1}{\pi P_n} \int_0^\pi \psi(t) \left(\sum_{k=0}^n p_{n-k} k \sin kt \right) dt \\ &\quad - \frac{l}{\pi P_n} \sum_{k=0}^n p_{n-k} (-1)^k. \end{aligned}$$

Since $\{p_n\}$ is nonnegative and nonincreasing and $P_n \rightarrow \infty$ as $n \rightarrow \infty$, we have by an application of Abel's Lemma

$$t_n(x) - \frac{l}{\pi} = o(1) + \frac{1}{\pi P_n} \int_0^\pi \psi(t) \left(\sum_{k=0}^n p_{n-k} k \sin kt \right) dt.$$

Denoting the $(C, 1) \cdot (N, p_n)$ mean of $\{nB_n(x)\}$ by $t_n^1(x)$, we write

$$t_n^1(x) - \frac{l}{\pi} = o(1) + \frac{1}{\pi(n+1)} \int_0^\pi \psi(t) \left\{ \sum_{r=0}^n \frac{1}{P_r} \sum_{k=0}^r p_{r-k} k \sin kt \right\} dt$$

by the regularity of the $(C, 1)$ mean. Thus, in order to prove the theorem, it is sufficient to show that as $n \rightarrow \infty$

$$(5.1) \quad I \equiv \frac{1}{n+1} \int_0^\pi \psi(t) g(n, t) dt = o(1),$$

where

$$(5.2) \quad g(n, t) = \sum_{r=0}^n \frac{1}{P_r} \sum_{k=0}^r p_k (\nu - k) \sin(\nu - k)t.$$

For a sufficiently small δ , such that $0 < \delta \leq \pi$, let us write

$$\begin{aligned} (5.3) \quad I &= \frac{1}{n+1} \left(\int_0^{n^{-1}} + \int_{n^{-1}}^\delta + \int_\delta^\pi \right) \psi(t) g(n, t) dt \\ &= I_1 + I_2 + I_3, \quad \text{say.} \end{aligned}$$

Since

$$(5.4) \quad |g(n, t)| \leq \sum_{r=0}^n \frac{1}{P_r} \sum_{k=0}^r p_k(v-k) \leq Kn^2,$$

we have by the hypothesis (3.1)

$$(5.5) \quad I_1 = O\left(n \int_0^{1/n} |\psi(t)| dt\right) = o(1/P_n) = o(1) \quad \text{as } n \rightarrow \infty.$$

By virtue of Lemma 3 for the interval $\delta \leq t \leq \pi$, we have

$$(5.6) \quad \frac{1}{n} g(n, t) = O\left(\frac{1}{n}\right) + O\left(\frac{1}{n} \sum_{r=0}^n \frac{1}{P_r}\right) + O\left(\frac{1}{P_n}\right) = o(1),$$

$n \rightarrow \infty,$

since the $(C, 1)$ mean is regular and $P_n \rightarrow \infty$ as $n \rightarrow \infty$. Thus, as $n \rightarrow \infty$

$$(5.7) \quad I_3 = \frac{1}{n} \int_{\delta}^{\pi} \psi(t) g(n, t) dt = o(1).$$

Finally, to prove that $I_2 = o(1)$, $n \rightarrow \infty$, we observe that by Lemma 3

$$(5.8) \quad \begin{aligned} I_2 &= O\left(\frac{1}{n} \int_{n^{-1}}^{\delta} t^{-2} |\psi(t)| dt\right) \\ &\quad + O\left(\frac{1}{n} \int_{n^{-1}}^{\delta} \frac{|\psi(t)|}{t} \left\{P_r \sum_{r=1}^n \frac{1}{P_r}\right\} dt\right) \\ &\quad + O\left(\frac{1}{P_n} \int_{n^{-1}}^{\delta} \frac{|\psi(t)|}{t} P_r dt\right) \\ &= O(I_{21}) + O(I_{22}) + O(I_{23}), \quad \text{say.} \end{aligned}$$

Integrating by parts, we have as $n \rightarrow \infty$

$$(5.9) \quad \begin{aligned} I_{21} &= \frac{1}{n} [t^{-2} \Psi(t)]_{n^{-1}}^{\delta} + \frac{2}{n} \int_{n^{-1}}^{\delta} \Psi(t) t^{-3} dt \\ &= o(1/P_n) + o(1) + o\left(\frac{1}{n} \int_{n^{-1}}^{\delta} t^{-2} \frac{1}{P_r} dt\right) \\ &= o(1) + o\left(\frac{1}{n} \int_{n^{-1}}^{\delta} t^{-2} dt\right) = o(1), \end{aligned}$$

by the hypothesis (3.1).

Let $V(n, r) = P_r \sum_{k=r}^n (1/P_k)$ and $n^{-1} < m^{-1} \leq \delta < (m-1)^{-1}$, then we write¹

$$(5.10) \quad \begin{aligned} I_{22} &= \frac{1}{n} \left(\sum_{r=m}^{n-1} \int_{(r+1)^{-1}+0}^{r^{-1}} + \int_{m^{-1}}^{\delta} \right) \frac{|\psi(t)|}{t} V(n, \tau) dt \\ &= I_{221} + I_{222}, \text{ say.} \end{aligned}$$

In the interval $(r+1)^{-1} < t \leq r^{-1}$, we have $V(n, \tau) = V(n, r)$, and therefore integrating by parts we get

$$(5.11) \quad \begin{aligned} &\int_{(r+1)^{-1}+0}^{r^{-1}} V(n, \tau) \frac{|\psi(t)|}{t} dt \\ &= V(n, r) \left[\frac{\Psi(t)}{t} \right]_{(r+1)^{-1}}^{r^{-1}} + \int_{(r+1)^{-1}+0}^{r^{-1}} V(n, \tau) \frac{\Psi(t)}{t^2} dt. \end{aligned}$$

Further, by (3.1) we have

$$(5.12) \quad \begin{aligned} &\sum_{r=m}^{n-1} V(n, r) \left\{ r\Psi\left(\frac{1}{r}\right) - (r+1)\Psi\left(\frac{1}{r+1}\right) \right\} \\ &= \sum_{r=m}^{n-1} \Delta_r \left\{ r\Psi\left(\frac{1}{r}\right) V(n, r) \right\} \\ &\quad - \sum_{r=m}^{n-1} (r+1)\Psi\left(\frac{1}{r+1}\right) \Delta_r V(n, r) \\ &= m\Psi\left(\frac{1}{m}\right) V(n, m) - nV(n, n)\Psi\left(\frac{1}{n}\right) \\ &\quad + o \left[\sum_{r=m}^{n-1} \frac{p_{r+1}}{P_{r+1}} \sum_{k=r}^n \frac{1}{P_k} - \sum_{r=m}^{n-1} \frac{1}{P_r} \right] \\ &= o \left(\sum_{r=m}^n \frac{1}{P_r} \right) + o(1) + o \left[\sum_{k=m}^n \frac{1}{P_k} \sum_{r=m}^k \frac{p_{r+1}}{P_{r+1}} \right] = o(n), \end{aligned}$$

since $(n+1)p_n \leq P_n$ and $\log n = O(P_n)$.

Combining (5.11) and (5.12), we obtain

$$(5.13) \quad I_{221} = o(1) + \int_{n^{-1}}^{m^{-1}} \frac{\Psi(t)}{t^2} \frac{1}{n} V(n, \tau) dt.$$

Similarly, it follows that

¹ Here m is an integer.

$$(5.14) \quad I_{222} = o(1) + \int_{n^{-1}}^{\delta} \frac{\Psi(t)}{t^2} \frac{1}{n} V(n, \tau) dt$$

since $(1/(n+1)) \sum_{k=m}^n (1/P_k) = o(1)$ by virtue of the regularity of the $(C, 1)$ mean and the fact that $P_n \rightarrow \infty$ as $n \rightarrow \infty$.

Thus, we have from (5.13) and (5.14)

$$(5.15) \quad \begin{aligned} I_{22} &= o(1) + \frac{1}{n} \int_{n^{-1}}^{\delta} t^{-2} \Psi(t) V(n, \tau) dt \\ &= o \left[\frac{1}{n} \int_{n^{-1}}^{\delta} \frac{1}{t} \left(\sum_{r=\tau}^n \frac{1}{P_r} \right) dt \right] \text{ by (3.1)} \\ &= o \left[\frac{1}{n} \int_{\delta^{-1}}^n \frac{1}{y} \left(\sum_{r=[y]}^n \frac{1}{P_r} \right) dy \right] = o(1), \quad n \rightarrow \infty, \end{aligned}$$

since

$$\sum_{k=1}^n \frac{1}{k} \sum_{r=k}^n \frac{1}{P_r} = \sum_{r=1}^n \frac{1}{P_r} \sum_{k=1}^r \frac{1}{k} = O(n),$$

by the hypothesis $\log n = O(P_n)$.

Following a technique parallel to that used in the proof of $I_{22} = o(1)$, we observe that

$$(5.16) \quad I_{23} = o(1) + \frac{1}{P_n} \int_{n^{-1}}^{\delta} t^{-2} \Psi(t) P_r dt = o \left(\frac{1}{P_n} \int_{n^{-1}}^{\delta} t^{-1} dt \right) = o(1),$$

$n \rightarrow \infty,$

by (3.1) and that $\log n = O(P_n)$.

Combining (5.9), (5.15) and (5.16) we get $I_2 = o(1)$, $n \rightarrow \infty$, which together with (5.5) and (5.7) implies (5.1).

This completes the proof of the theorem.

6. Proof of Theorem 2. As in the proof of Theorem 1, in order to prove Theorem 2 it is sufficient to show that as $n \rightarrow \infty$

$$(6.1) \quad I_k = o(1) \quad (k = 1, 2, 3),$$

under the hypotheses of Theorem 2.

Since by (5.4), $(1/n) g(n, t) = O(n)$ and $\psi(t) = o(1)$, $t \rightarrow 0$, we have as $n \rightarrow \infty$

$$(6.2) \quad I_1 = o(1).$$

From (5.6) it follows that as $n \rightarrow \infty$

$$(6.3) \quad I_3 = o(1).$$

By Lemma 3, we have

$$(6.4) \quad \begin{aligned} \frac{1}{n} \int_{n^{-1}}^{\delta} |g(n, t)| dt &\leq \frac{K}{n} \int_{n^{-1}}^{\delta} t^{-2} dt \\ &+ \frac{K}{n} \int_{n^{-1}}^{\delta} \frac{P(1/t)}{t} \left\{ \sum_{\nu=r}^n \frac{1}{P_{\nu}} \right\} dt + \frac{K}{P_n} \int_{n^{-1}}^{\delta} \frac{P(1/t)}{t} dt \\ &\leq K + \frac{K}{n} \int_{\delta^{-1}}^n \frac{P(y)}{y} \left\{ \sum_{\nu=[y]}^n \frac{1}{P_{\nu}} \right\} dy + \frac{K}{P_n} \int_{\delta^{-1}}^n \frac{P(y)}{y} dy \\ &\leq K + \frac{K}{n} \int_{\delta^{-1}}^n \frac{P(y)}{y} \left\{ \sum_{\nu=[y]}^n \frac{1}{P_{\nu}} \right\} dy \leq K, \end{aligned}$$

since $S_n = O(1)$ and

$$\frac{1}{n} \sum_{k=1}^n \frac{P_k}{k} \sum_{\nu=k}^n \frac{1}{P_{\nu}} = \frac{1}{n} \sum_{\nu=1}^n \frac{1}{P_{\nu}} \sum_{k=1}^{\nu} \frac{P_k}{k} \leq \frac{1}{n} \sum_{\nu=1}^n S_{\nu} \leq K.$$

Finally, since $\psi(t) = o(1)$, $t \rightarrow 0$, it follows from (6.4) that

$$(6.5) \quad I_2 = \frac{1}{n} \int_{n^{-1}}^{\delta} \psi(t) g(n, t) dt = o(1) \quad \text{as } n \rightarrow \infty.$$

Combining (6.2), (6.3) and (6.5), we prove (6.1), which completes the proof of Theorem 2.

7. It may be remarked that the hypothesis $S_n = O(1)$ of our Theorem 2 has its origin in [2], where its intrinsic character has been discussed by Hille and Tamarkin.

By making use of a Tauberian result due to Iyengar [3], Varshney has pointed out in [8] that under the condition (2.1) of Theorem A $(N, 1/(n+1)) \cdot (C, 1)$ summability of $\{nB_n(x)\}$ implies its $(C, 1)$ summability. In view of this, Theorem A and our Theorem 1 suggest the open question of $(C, 1) \cdot (N, 1/(n+1))$ summability of $\{nB_n(x)\}$ implying its $(C, 1)$ summability under the condition (2.1).

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