COBORDISM PROPERTIES OF MANIFOLDS OF SMALL CATEGORY

M. V. MIELKE

Cobordism properties of certain manifolds M with $C(M) \leq 3$ ($C(M)$ is the Lusternik-Schnirelman category of M) are studied (Lemma 1) with an application to "killing" homotopy groups by surgery (Theorem 1*).

An n -manifold is a smooth, connected, compact, n -dimensional manifold without boundary. " $[]$ " denotes the "greatest integer" function.

THEOREM 1. *If M is a $[n/3]$ -connected n -manifold where $n \equiv 3 \pmod{4}$ then M cobounds the n -sphere S^n .*

An equivalent statement of Theorem 1 is:

THEOREM 1*. *If the homotopy groups of an n -manifold M ($n \equiv 3 \pmod{4}$) can be killed by spherical modifications (surgery) through dimension $[n/3]$ then all the homotopy groups through dimension $n-1$ can be so killed (cf. [5]).*

PROOF OF THEOREM 1. By [1, p. 52, 199-200] any $[n/3]$ -connected n -manifold ($n > 3$) satisfies $C(M) \leq 3$. In view of this and the fact that any 3-manifold cobounds S^3 (cf. [4, p. 81]) the theorem follows from Lemma 1.

LEMMA 1. *Let M be an n -manifold with $C(M) \leq 3$. If $n \equiv 3 \pmod{4}$ then M cobounds S^n .*

PROOF OF LEMMA 1. In view of [4] we may assume $n > 3$. Let $W_i \in H^i(M; \mathbb{Z}_2)$ be the i th Stiefel-Whitney class of M . The Stiefel-Whitney number associated to a tuple $i_1, \dots, i_r$ where $i_1 + \dots + i_r = n$ is defined to be $\langle W_{i_1} \cdots W_{i_r}, \mu \rangle$ where " $\langle \rangle$ " denotes the Kronecker index and $\mu \in H_n(M; \mathbb{Z}_2)$ is the fundamental class of M (cf. [2]). By [4, p. 77] M cobounds S^n if and only if all Stiefel-Whitney numbers of M are zero. To show M cobounds S^n it is sufficient, then, to show $W_{i_1} \cdots W_{i_r} = 0$ for all tuples $i_1, \dots, i_r$ satisfying $i_1 + \dots + i_r = n$.

A well-known result ([3, p. 4]) on cup products is:

1. If $C(M) \leq 3$ then the cup product of 3 or more positive dimensional cohomology classes is zero.

In view of this, Lemma 1 will be proved when it is shown that $W(i) = 0$ for $i = 0, \dots, n$ where $W(i) = W_i W_{n-i}$.

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The following results are needed for the proof. According to Wu [6] there exist cohomology classes $U_r \in H^r(M; \mathbb{Z}_2)$ ($r=0, \dots, n$) satisfying:

2. $U_i = W_i + \sum_{r=0}^{i-1} \text{Sq}^{i-r}(U_r)$ where $U_0 = W_0 = 1$ and "Sq" is the Steenrod square [3].

3. $U_r P_{n-r} = \text{Sq}^r(P_{n-r})$ where P_{n-r} is a homogeneous polynomial of degree $n-r$ in the Stiefel-Whitney classes.

The Steenrod squares satisfy the Cartan formula ([3]).

4. $\text{Sq}^r(W_p W_q) = \sum_{a=0}^r \text{Sq}^a(W_p) \text{Sq}^{r-a}(W_q)$.

In Wu [7] it is shown that:

5. $\text{Sq}^r(W_i) = \sum_{t=0}^i B(i-r+t, t) W_{r-t} W_{i+t}$ for $r \leq i$ and zero otherwise. $B(X, t)$ is the binomial coefficient $\binom{X}{t} \pmod{2}$ if $X-1 \geq t$ and 0 otherwise.

For $X \geq 1$ an easy computation shows that:

6. $B(X, 0) = 1$ all integers X and $B(X, i) = 1$ if and only if $X \equiv 0, i+1 \pmod{4}$ for $i=1, 2, 3$.

Let $P_{n-r} = W_p W_q$ in $3(p+q=n-r)$. A simple calculation using 1, 3, 4, 5 yields:

7. If $n=p+q+r$, $0 < m = \min(p, r)$ and $r \leq q$ then $\sum_{a=0}^m B(p, a) \cdot B(q, r-a) W(p+a) = 0$.

LEMMA 2. Let M be an n -manifold with $C(M) \leq 3$. If $n=4s+3$ then:

- $W(4t+1) = 0$ ($0 \leq t \leq s-1$)
- $W(4t+2) = 0$ ($0 \leq t \leq s-1$)
- $W(4t) + W(4t+1) = 0$ ($1 \leq t \leq s$)
- $\sum_{i=0}^3 W(4t+i) = 0$ ($1 \leq t \leq s-1$)
- $W(0) = 0$ ($s \geq 0$).

PROOF OF LEMMA 2.

a. Let $r=2$ and $p=4t+1$. The conditions on n, t and $n=p+q+r$ imply $0 < p \equiv 1$ and $0 < q \equiv 0 \pmod{4}$. The result follows from 6 and 7.

b. Let $r=2$ and $p=4t+2$. The conditions on n and t imply $0 < p \equiv 2$ and $0 < q \equiv 3 \pmod{4}$. The result follows from 6 and 7.

c. Let $r=1$ and $p=4t$. The conditions on n and t imply $0 < p \equiv 0$ and $0 < q \equiv 2 \pmod{4}$. 6 and 7 give the result.

d. Let $r=3$ and $p=4t$. The conditions on n and t imply $0 < p \equiv 0$ and $0 < q \equiv 0 \pmod{4}$. 6 and 7 give the result.

e. From 2, $W_0 = U_0 = 1$ and $W_1 = U_1$. Letting $r=1$ and $P_{n-r} = W_{n-1}$ in 3 gives $W_1 W_{n-1} = \text{Sq}^1(W_{n-1})$. Using 5 and 6 with $n-1 \equiv 2 \pmod{4}$ gives $W(0) = 0$. This proves Lemma 2.

Let $s \geq 1$. Letting $t=0$ in a and b, $t=s$ and $W(i) = W(n-i)$ ($i=1, 2$) in c, and e yields:

8. $W(i) = 0, i=0, 1, 2, 3$.

This proves Lemma 1 for $n=7$.

If $s \geq 2$ then $t=1, \dots, s-1$ in a, b, c and d gives:

9. $W(i)=0, i=4, 5, \dots, n-4$. For $n \geq 11$ Lemma 1 follows from 8 and 9. This completes the proof of Lemma 1 and hence of the theorem.

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