ALL ALGEBRAS OF SPHERICAL FUNCTIONS DEFINED ON THE TWO-BY-TWO GENERAL LINEAR GROUP WITH ENTRIES IN A LOCALLY COMPACT p -ADIC FIELD ARE COMMUTATIVE¹

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1. Introduction. Let Ω be a locally compact p -adic field, with residue class field of odd characteristic, let $\mathfrak{o}$ denote the ring of integers in Ω , and let τ generate the unique prime ideal in $\mathfrak{o}$, i.e. $\text{ord}(\tau) = 1$. Let $G = GL(2, \Omega)$, the group of nonsingular two-by-two matrices with entries in Ω , and let $K = GL(2, \mathfrak{o})$, the subgroup of integer matrices with unit determinant. K is a maximal compact subgroup of G , and any maximal compact subgroup of G is conjugate to K .

Let u be an irreducible unitary representation of K in a complex vector space V_u . Let S_u denote the vector space consisting of functions f on G whose values are endomorphisms of V_u and which satisfy $f(kgk') = u(k)f(g)u(k')$ for all $k, k' \in K, g \in G$ (the product is composition of endomorphisms of V_u). S_u is called the space of spherical functions of class u on G . Set $f^*(g) = {}^t\bar{f}(g^{-1})$. Those elements of S_u which are integrable (for the Haar measure) on $G(S_u^1)$ are seen to comprise a selfadjoint Banach algebra under convolution on G .

In this paper we shall prove that, for any u , S_u^1 is a commutative algebra. We remark that our result implies the commutativity of a class of Hecke algebras defined on certain two-by-two matrix groups; it also implies that $GL(2, \Omega)$ is a type I group (cf. [1] for an earlier proof that $GL(2, \Omega)$ is a type I group).

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2. A reduction. Let ϵ be a nonsquare unit in $\mathfrak{o}$. Let R be the subset of K consisting of matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \epsilon\beta & \alpha \end{pmatrix}.$$

R is a compact abelian group and is isomorphic to the group of units in the extension ring $\mathfrak{o}(\sqrt{\epsilon})$.

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Let ρ be a character of R , A_ρ the convolution algebra of integrable functions on G satisfying the relation $f(rgr') = \rho(rr')f(g)$ for all $r, r' \in R$, $g \in G$. It is easy to show that S_u^1 is isomorphic to a subalgebra of A_ρ for any ρ contained in the restriction to R of u (cf. [2, p. 173]). To prove S_u^1 is commutative for any u we show that A_ρ is commutative for any ρ .

3. The proof. To facilitate the proof of the commutativity of the algebras A_ρ we introduce a "Cayley transform" for G .

For $g \in G$ set $C(g) = \gamma g \gamma^{-1}$, where

$$\gamma = \begin{pmatrix} 1 & -1/\sqrt{\epsilon} \\ \sqrt{\epsilon} & 1 \end{pmatrix}.$$

C is an isomorphism of G with the group of all nonsingular matrices of the form

$$\begin{pmatrix} a & b \\ -\epsilon \bar{b} & \bar{a} \end{pmatrix},$$

where $a, b \in \Omega(\sqrt{\epsilon})$ and the bar denotes conjugation in $\Omega(\sqrt{\epsilon})$. $C(K) = C(G) \cap GL(2, \mathfrak{D}(\sqrt{\epsilon}))$; $C(R)$ is the set of diagonal matrices in $C(K)$. From now on we identify $C(G)$ and G .

In the transformed version of G we consider two involutions. The mapping g to $\bar{g}$, where $\bar{g}$ has entries conjugate to those of g , is an order-two automorphism of G . For

$$\bar{g} = \begin{pmatrix} a & b \\ -\epsilon \bar{b} & \bar{a} \end{pmatrix},$$

$0 \neq \Delta(g)$, the determinant of g , set

$$g^\sigma = \Delta^{-1}(g) \begin{pmatrix} \bar{a} & b \\ -\epsilon \bar{b} & a \end{pmatrix}.$$

σ is an order two antiautomorphism of G . Both involutions clearly leave a Haar measure on G invariant.

We shall say g is σ -symmetric if $g^\sigma = g\Delta^{-1}(g)$. This is the case if and only if diagonal entries of g belong to Ω . Note that $g^\sigma = g^{-1}$ if and only if g is a diagonal matrix (a scalar times an element of R). We denote the σ -symmetric elements of G by S . $\triangleleft$

LEMMA 1. (1) $G = SR$.

(2) $rsr' \in S$, $s \in S$, $r, r' \in R$ if and only if, diagonal entries of s are zero or $r' = \lambda r^{-1}$, λ a scalar matrix with unit entries in $\mathfrak{D}$.

(3) For all $f \in A_\rho$ and all $s \in S$, $f(s) = f(\bar{s})$.

Proof of (3) only. For any $a \in \Omega$, unit $b \in \mathcal{O}(\sqrt{\epsilon})$, and integer k we have

$$\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a & b\tau^k \\ -\epsilon b\tau^k & a \end{pmatrix} = \begin{pmatrix} a & b\tau^k \\ -\epsilon b\tau^k & a \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}.$$

(3) follows immediately.

Next we induce an involution σ on A_p and show that the σ -symmetric elements of A_p form a commutative real subalgebra.

LEMMA 2. For $f \in A_p$, set $f^\sigma(g) = \bar{f}(g^\sigma)$. Then $f^\sigma \in A_p$ and the mapping $\sigma: A_p \rightarrow A_p$ is a complex-conjugate linear antiautomorphism.

PROOF. $f \in A_p$ implies $f^\sigma \in A_p$, since $f^\sigma(rgr') = \bar{f}((rgr')^\sigma) = \bar{f}(r'^\sigma g^\sigma r^\sigma) = \bar{f}(r'^{-1} g^\sigma r^{-1}) = \rho(rr') \bar{f}(g^\sigma)$, for any $r, r' \in R, g \in G$. That σ is complex-conjugate linear is clear. To see that σ is an antiautomorphism observe that

$$\begin{aligned} (f * h)^\sigma(g) &= \int_G \bar{f}(x) \bar{h}(x^{-1}g^\sigma) dx = \int_G \bar{f}(g^\sigma u^{-1}) \bar{h}(u) du \\ &= \int_G \bar{h}(u^\sigma) \bar{f}(g^\sigma (u^{-1})^\sigma) du = \int_G h^\sigma(u) f^\sigma(u^{-1}g) du = h^\sigma * f^\sigma(g). \end{aligned}$$

Call $f \in A_p$ σ -symmetric if $f^\sigma(g) = f(g)$ for all $g \in G$.

LEMMA 3. (1) $f \in A_p$ is σ -symmetric if and only if $f(g) = \bar{f}(\Delta^{-1}(g)\bar{g})$.

(2) The σ -symmetric elements of A_p comprise a commutative real subalgebra of A_p .

PROOF. (1) $f^\sigma(g) = f^\sigma(sr) = \bar{f}(r^\sigma s^\sigma) = \bar{f}(r^{-1}\Delta^{-1}(s)\bar{s}) = \rho(r)\bar{f}(\Delta^{-1}(s)\bar{s}) = \rho^{-1}(\bar{r})\rho(\Delta(r))\bar{f}(\Delta^{-1}(s)\bar{s}) = \bar{f}(\Delta^{-1}(g)\bar{g})$.

(2) Clearly, real linear combinations of σ -symmetric elements of A_p are σ -symmetric. We must show both that $f * h(s) = \bar{f} * \bar{h}(\Delta^{-1}(s)\bar{s})$ and that $f * h(s) = h * f(s)$ for any f and h σ -symmetric and any $s \in S$. We have

$$\begin{aligned} f * h(s) &= \int_G f(g) h(g^{-1}s) dg \\ &= \int_G \bar{f}(\Delta^{-1}(g)\bar{g}) \bar{h}(\Delta(g)\bar{g}^{-1}\Delta^{-1}(s)\bar{s}) dg \\ &= \int_G \bar{f}(x) \bar{h}(x^{-1}\Delta^{-1}(s)\bar{s}) dx = \bar{f} * \bar{h}(\Delta^{-1}(s)\bar{s}); \end{aligned}$$

and $f * h(s) = (f * h)^\sigma(s) = h^\sigma * f^\sigma(s) = h * f(s)$.

PROPOSITION. A_p is commutative.

PROOF. Given $f \in A_p$, we may write $f = f_1 + if_2$, where f_i is σ -symmetric, $i = 1$ or 2 . Indeed, $f_1 = (f + f^\sigma)/2$ and $f_2 = (f - f^\sigma)/2i$. The commutativity of A_p follows immediately from Lemma 3.

4. **Extension of results to the projective general linear group.** Our results extend immediately, by a lifting argument, to the case of $G = PGL(2, \Omega)$, the factor group of $GL(2, \Omega)$ by the scalar matrices, and $K = PGL(2, \mathfrak{O})$. In $PGL(2, \Omega)$ there is a second conjugacy class of maximal compact subgroups, a representative of which is generated by the projections to $PGL(2, \Omega)$ of the matrices

$$\left\{ k = \begin{pmatrix} k_{11} & k_{12} \\ \tau k_{21} & k_{22} \end{pmatrix} : \begin{matrix} k_{ij} \in \mathfrak{O}, & 1 \leq i, j \leq 2 \\ \Delta^{-1}(k) \in \mathfrak{O} \end{matrix} \right\} \cup \left\{ \begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix} \right\}.$$

This group contains a compact abelian subgroup isomorphic to $\Omega^\times(\sqrt{\tau})/\Omega^\times$, the quotient of multiplicative groups. A variant of the argument of this paper extends the conclusions.

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