HILBERT-SCHMIDT REPRESENTATIONS OF GROUPS

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Introduction. A *completely continuous representation* of a locally compact group G is a unitary representation T of G for which the operator T_b is completely continuous whenever b is an element of the group C^* -algebra $C^*(G)$. We have also that, with respect to the norm operator topology, $T_{[L(G)]}$ is dense in the set $T_{[L^1(G)]}$ which in turn is dense in the set $T_{[C^*(G)]}$. (Here " $L(G)$ " is the linear space of all continuous complex-valued functions on G with compact support.) Further, it is known that, if T is an irreducible completely continuous representation of G , the set $T_{[C^*(G)]}$ is precisely the algebra of all completely continuous operators on the Hilbert space $H(T)$. (See [4].) Investigation of completely continuous representations has been fruitful because, first of all, such representations are more easily analyzed, and secondly, a large class of groups (CCR-groups [2]) satisfies the condition that each of their irreducible unitary representations is a completely continuous representation.

The algebra A of all Hilbert-Schmidt operators on a Hilbert space is dense in the algebra of all completely continuous operators on that space. Therefore, it was natural to inquire what simplifications arise if one requires that a completely continuous representation of a group G should map either the set $L(G)$ into the space of all Hilbert-Schmidt operators or the set $L^1(G)$ into the space of all Hilbert-Schmidt operators. In this paper we show that too much simplification occurs in the latter case. In fact, if T_f is a Hilbert-Schmidt operator whenever f is an element of $L^1(G)$, then T must be finite dimensional. (See the theorem below.)

In the first case mentioned above, the simplifications, if there are any, are more elusive, and we shall return to this question at the end of the article.

DEFINITION 1. Let G be a locally compact group. By a *unitary representation* of G we mean a homomorphism T of G into the group of unitary operators on a Hilbert space $H(T)$ which is continuous with respect to the weak-operator topology. (See [3].)

We single out this definition from representation theory because, in the course of the proof below, the continuity condition here is heavily exploited.

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DEFINITION 2. Let H be a Hilbert space and let $[v_a]$ be an orthonormal basis for H . An operator T on H is called a *Hilbert-Schmidt operator* if $\sum_a \|T(v_a)\|^2$ is finite. (See [1].)

THEOREM. Let T be a unitary representation of a locally compact group G for which T_f is a Hilbert-Schmidt operator whenever f is an element of $L^1(G)$. Then T is finite dimensional.

PROOF. Assume that $[v_n]$ is an infinite orthonormal sequence in $H(T)$. With this assumption, we shall define a sequence $[(n_k, w_k, U_k, f_k)]$ of quadruples, where $[n_k]$ is an increasing sequence of positive integers, $w_k = v_{(n_k)}$, $[U_k]$ is a sequence of neighborhoods of the identity e in G , and f_k is a multiple of the characteristic function $\chi_{(U_k)}$ of the neighborhood U_k . We will then show that $f = \sum_{k=1}^{\infty} f_k$ is an element of $L^1(G)$ such that T_f is not a Hilbert-Schmidt operator.

Thus let ϵ be a positive number which is less than one, and, for each positive integer k , define a_k to equal $(1/k)^{1/2} - (1/(k+1))^{1/2}$.

Let $n_1 = 1$, $w_1 = v_1$, U_1 be a neighborhood of e with finite measure such that, for each element y of U_1 ,

$$re(T_y(w_1), w_1) > \epsilon,$$

and $f_1 = (a_1/\lambda(U_1))\chi_{(U_1)}$. (We denote Haar measure by λ .)

Now if $(n_1, w_1, U_1, f_1) \cdots (n_k, w_k, U_k, f_k)$ have been defined, we proceed as follows to define $(n_{(k+1)}, w_{(k+1)}, U_{(k+1)}, f_{(k+1)})$. Let $n_{(k+1)}$ be an integer which is greater than n_k , and for which

$$\|T_{[\sum_{i=1}^k f_i]}(v_n)\| < \epsilon/(k+1)$$

for each integer $n \geq n_{(k+1)}$. Such an integer exists because $T_{[\sum_{i=1}^k f_i]}$ is by assumption a Hilbert-Schmidt operator and consequently is completely continuous.

Define $w_{(k+1)}$ to be $v_{(n_{(k+1)})}$; define $U_{(k+1)}$ to be the intersection of U_k with the set of all elements y in G such that

$$re(T_y(w_{(k+1)}), w_{(k+1)}) > \epsilon;$$

finally define $f_{(k+1)}$ to be $(a_{(k+1)}/\lambda(U_{(k+1)}))\chi_{(U_{(k+1)})}$. This completes our inductive definition.

REMARK 1. For each positive integer k , $\|f_k\|_1 = a_k$. Hence $f = \sum_{k=1}^{\infty} f_k$ is an element of $L^1(G)$, and $\|f\|_1 = 1$. Notice also that $\sum_{k=n}^{\infty} a_k = (1/(n+1))^{1/2}$.

REMARK 2. If k and n are positive integers such that $k > n$, and if y is an element of U_k , then y is also an element of U_n and $re(T_y(w_n), w_n) > \epsilon$.

We now compute to see that T_f , where $f = \sum_{k=1}^{\infty} f_k$, is not a Hilbert-Schmidt operator on $H(T)$. Thus let $[v_a]$ be an extension of $[v_n]$ to an orthonormal basis for $H(T)$. Then:

$$\begin{aligned}
 \sum_a \|T_f(v_a)\|^2 &\geq \sum_n \|T_f(v_n)\|^2 \geq \sum_n \|T_f(w_n)\|^2 \\
 &\geq \sum_n |(T_f(w_n), w_n)|^2 = \sum_n |(T_{[\sum_{k=1}^{\infty} f_k]}(w_n), w_n)|^2 \\
 &= \sum_n |(T_{[\sum_{k=1}^{n-1} f_k]}(w_n), w_n) + (T_{[\sum_{k=n}^{\infty} f_k]}(w_n), w_n)|^2 \\
 &\geq \sum_n | |(T_{[\sum_{k=n}^{\infty} f_k]}(w_n), w_n)| - |(T_{[\sum_{k=1}^{n-1} f_k]}(w_n), w_n)| |^2 \\
 &= \sum_n | |(T_{[\sum_{k=n}^{\infty} f_k]}(w_n), w_n)|^2 + |(T_{[\sum_{k=1}^{n-1} f_k]}(w_n), w_n)|^2 \\
 &\quad - 2 |(T_{[\sum_{k=n}^{\infty} f_k]}(w_n), w_n)| |(T_{[\sum_{k=1}^{n-1} f_k]}(w_n), w_n)| | \\
 &\geq \sum_n | |(T_{[\sum_{k=n}^{\infty} f_k]}(w_n), w_n)|^2 + |(T_{[\sum_{k=1}^{n-1} f_k]}(w_n), w_n)|^2 \\
 &\quad - 2 |(T_{[\sum_{k=n}^{\infty} f_k]}(w_n), w_n)| |(T_{[\sum_{k=1}^{n-1} f_k]}(w_n), w_n)| | \\
 &\geq \sum_n \left[\left| \sum_{k=n}^{\infty} (T_{f_k}(w_n), w_n) \right|^2 - \|T_{[\sum_{k=1}^{n-1} f_k]}(w_n)\|^2 \right. \\
 &\quad \left. - 2 |(T_{[\sum_{k=n}^{\infty} f_k]}(w_n), w_n)| \|T_{[\sum_{k=1}^{n-1} f_k]}(w_n)\| \right] \\
 &\geq \sum_n \left[\left[\operatorname{re} \left[\sum_{k=n}^{\infty} (T_{f_k}(w_n), w_n) \right] \right]^2 - \epsilon^2/n^2 - 2 \sum_{k=n}^{\infty} \|f_k\|_{1\epsilon/n} \right] \\
 &= \sum_n \left[\left[\sum_{k=n}^{\infty} \operatorname{re}(T_{f_k}(w_n), w_n) \right]^2 - \epsilon^2/n^2 - 2(\epsilon/n(n+1)^{1/2}) \right]
 \end{aligned}$$

(see Remark 1)

$$\begin{aligned}
 &= \sum_n \left[\left[\sum_{k=n}^{\infty} \operatorname{re} \int_G f_k(y) (T_y(w_n), w_n) d\lambda(y) \right]^2 \right. \\
 &\quad \left. - \epsilon^2/n^2 - 2(\epsilon/n(n+1)^{1/2}) \right] \\
 &= \sum_n \left[\left[\sum_{k=n}^{\infty} (a_k/\lambda(U_k)) \operatorname{re} \int_{(U_k)} (T_y(w_n), w_n) d\lambda(y) \right]^2 \right. \\
 &\quad \left. - \epsilon^2/n^2 - 2(\epsilon/n(n+1)^{1/2}) \right] \\
 &\geq \sum_n \left[\epsilon^2 \left[\sum_{k=n}^{\infty} a_k \right]^2 - \epsilon^2/n^2 - 2(\epsilon/n(n+1)^{1/2}) \right]
 \end{aligned}$$

(see Remark 2)

$$= \sum_n [\epsilon^2/(n+1) - \epsilon^2/n^2 - 2(\epsilon/n(n+1)^{1/2})] = \infty,$$

and this shows that T_f is not a Hilbert-Schmidt operator.

COROLLARY 1. *If T is an infinite dimensional unitary representation of a locally compact group G , then there must exist an element f of $L^1(G)$ for which the operator T_f has infinite dimensional range.*

DEFINITION 3. Let H be a Hilbert space. An operator T on H is said to have *finite trace* if there exist Hilbert-Schmidt operators S and S' on H such that $T = SS'$. (See [1].)

COROLLARY 2. *If a unitary representation T of a locally compact group G satisfies the condition that T_f has finite trace whenever f is an element of $L^1(G)$, then T is finite dimensional.*

PROOF. This follows from the theorem since the product of Hilbert-Schmidt operators is again a Hilbert-Schmidt operator.

COROLLARY 3. *If a unitary representation T of a locally compact group G satisfies T_b is a Hilbert-Schmidt operator whenever b is an element of $C^*(G)$, then T is finite dimensional.*

Of course this is well known in case T is irreducible. If T is not irreducible, the result is less obvious since one can construct a norm-closed subalgebra A of the algebra $B(H)$ of all bounded linear operators on an infinite dimensional Hilbert space H such that each element T of A has one dimensional range, hence is a Hilbert-Schmidt operator, but for which the union of the ranges of the elements of A is all of H . This corollary asserts that such a subalgebra of $B(H)$ cannot occur as the image of a group C^* -algebra.

We turn now to the question mentioned in the introduction of what happens if we require only that a unitary representation T of a locally compact group G carry each element f of $L(G)$ onto a Hilbert-Schmidt operator. The following propositions indicate the striking difference between this case and the one dealt with above. They demonstrate the existence of many infinite dimensional unitary representations which map $L(G)$ into the class of Hilbert-Schmidt operators.

PROPOSITION 1. *If G is a compact abelian group, R is the left regular representation of G , and f is an element of $L(G)$, then R_f is a Hilbert-Schmidt operator on $L^2(G)$.*

PROOF. If $[\chi_n]$ is the set of all characters on G , then $[\chi_n]$ is an orthonormal basis for $L^2(G)$. The proposition then follows because $f^*\chi_n = (f, \chi_n)\chi_n$.

PROPOSITION 2. *If T is an irreducible unitary representation of a simply connected semisimple Lie group, then T_f is a Hilbert-Schmidt operator whenever f is an element of $L(G)$.*

This proposition may be deduced from a theorem of Harish-Chandra. (See Theorem 5 of *Representations of a semisimple Lie group*. III, Trans. Amer. Math. Soc. **76** (1954), 234–253.)

PROPOSITION 3. *Let G be a unimodular group. If T is an irreducible subrepresentation of the left regular representation of G , then T_f is a Hilbert-Schmidt operator whenever f is an element of $L(G)$.*

This result is included in a paper which the author has submitted for publication.

We remark, finally, that results have been obtained concerning unitary representations which map “sufficiently” differentiable functions onto Hilbert-Schmidt operators. (See Theorem 4.2 of Nelson and Stinespring’s article *Representations of elliptic operators in an enveloping algebra*, Amer. J. Math **81** (1959), 547–560.)

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