SOME REMARKS ON STRUCTURE OF RIESZ OPERATORS

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1. Let H be a Hilbert space and $T \in \mathcal{L}(H)$, the Banach algebra of all bounded linear operators on H . By $\sigma(T)$ we denote the spectrum of T and $\eta_T(\lambda)$ denotes the subspace of all eigenvectors of the proper value λ .

In this note we give some conditions under which a Riesz operator is normal or compact.

2. At first we give

DEFINITION 2.1 [8]. A Riesz point of $\sigma(T)$ is a point λ of $\sigma(T)$ such that

$$H = N(\lambda; T) \oplus F(\lambda; T)$$

where $\dim N(\lambda; T) < \infty$ and $F(\lambda; T)$ is closed, $T(N) \subset N$, $T(F) \subset F$, and $\lambda I - T$ restricted to N is nilpotent while $\lambda I - T$ restricted to F is a homeomorphism.

T is a Riesz operator in H if $T \in \mathcal{L}(H)$ and if each point of $\sigma(T) - \{0\}$ is a Riesz point.

DEFINITION 2.2 [6]. An operator T is said to be hyponormal if $T^*T - TT^* \geq 0$.

LEMMA 2.1. If T is a Riesz operator and hyponormal then T is a normal operator.

PROOF. Since T is hyponormal it is known that $Tx = \lambda x$ implies $T^*x = \bar{\lambda}x$ and it follows that $\{\eta_T(\lambda) : \lambda \in \sigma(T)\}$ is a mutually orthogonal family of reducing subspaces. Then the restriction of T onto $H_0 = \sum_{\lambda \in \sigma(T)} \eta_T(\lambda)$ is a normal operator. To complete the proof, we have only to prove that the restriction $T_1 = T|_{H_1}$, where $H_1 = H_0^\perp$ is zero. Since H_1 is invariant under T it follows that T_1 is hyponormal and therefore exists a point $0 \neq \lambda \in \sigma(T)$ such that $|\lambda| = \|T_1\|$. This contradiction implies that T is normal.

A direct proof for this lemma follows from Theorem 3 of [6] since $\sigma(T)$ for Riesz operators have a single limit point and therefore it is normal.

DEFINITION 2.3 [2]. An operator T is said to be of class (N) in case $\|T^2x\| \geq \|Tx\|^2$ for all unit vectors $x \in H$.

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THEOREM 2.1. *If T is a Riesz operator of class (N) then T is a normal.*

PROOF. Since T is of class (N) it is normaloid [2]. If $\lambda \in \sigma(T)$ with $|\lambda| = \|T\|$ and $0 \neq x \in \eta_T(\lambda)$ then we have

$$\|T^*x - \bar{\lambda}x\|^2 = \|T^*x\|^2 - \bar{\lambda}\langle x, T^*x \rangle - \lambda\langle T^*x, x \rangle + |\lambda|^2\|x\|^2 \leq 0$$

and it follows that $\eta_T(\lambda) = \eta_{T^*}(\bar{\lambda})$. Since $\eta_T(\lambda)$ reduces T we consider $H_\lambda = \eta_T(\lambda)^\perp$; it follows that H_λ reduces T and $T|_{H_\lambda}$ is a Riesz operator of class (N) . For the same reason as above in the subspace H_λ we conclude that there exists $\lambda_2 \in \sigma(T|_{H_\lambda})$, $|\lambda_2| = \|T|_{H_\lambda}\|$ and such that $\eta_T(\lambda_2)$ reduces $T|_{H_\lambda}$. With this method we obtain that $\{\eta_T(\lambda) : 0 \neq \lambda \in \sigma(T)\}$ is a mutually orthogonal family of reducing subspaces. The rest of the proof is quite similar to that of Lemma 2.1.

COROLLARY 2.1. *If T is an operator of class (N) and $T^{*n}T^n$ is a Riesz operator for nonnegative integer n , then T is a normal operator.*

PROOF. Since $T^{*n}T^n$ is a selfadjoint Riesz operator, it follows (see [9], p. 742) that $T^{*n}T^n$ is compact and the result follows from Theorem 2 [2].

It is known that [9] a Riesz operator on a Hilbert space is decomposable, which means that $T = C + Q$ where C is compact and Q is a quasinilpotent operator.

COROLLARY 2.2. *If T is a nonzero Riesz operator on Hilbert space then its compact part $C \neq 0$.*

PROOF. Since T is of class (N) it is normaloid. If there exists a decomposition with its compact part $C = 0$ the corresponding T is a quasinilpotent normaloid operator and by [6] it follows that $T = 0$.

In [4] I. Kaplansky has proved that if A and B are normal compact operators with "property (L)" then A and B commute.

COROLLARY 2.3. *Let A and B be Riesz operators of class (N) with property (L), relative to the sequences $\{\lambda_i\}$ and $\{\mu_i\}$ for all $\mu_i \neq 0$. Then A and B commute.*

The proof is obvious since A and B are normal operators and by [9] they are compact operators.

LEMMA 2.2. *Let $T \in \mathcal{L}(H)$ which possesses a set of eigenvectors fundamental in H . If every invariant subspace under T reduces T , then T is a normal operator.*

PROOF. Since every invariant subspace under T reduces T , in particular each $\eta_T(\lambda)$, $\lambda \in \sigma_p(T)$ is a reducing subspace for each

$\lambda \in \sigma_p(T)$. For the same reason as in Lemma 2.1 we obtain that every eigenvector for T with eigenvalue λ is an eigenvector for T^* with eigenvalue $\bar{\lambda}$ and then $\{\eta_T(\lambda) : \lambda \in \sigma_p(T)\}$ is a mutually orthogonal family of reducing subspaces and the restriction of T onto $H_0 = \sum_{\lambda \in \sigma_p(T)} \oplus \eta_T(\lambda)$ is a normal operator. Since H is spanned by eigenvectors, T is a normal operator.

COROLLARY 2.4. *If T is a Riesz operator on a Hilbert space H spanned by eigenvectors of T and every subspace which is invariant under T reduces T , then T is compact.*

PROOF. From Lemma 2.2 the operator T is normal. Since T is Riesz and a normal operator it is compact.

3. In this section we give some remarks which were suggested by the results in [7], [3].

Let $I(H)$ be the two-sided ideal of compact operators. We note by $\mathcal{C}$ the Banach algebra $\mathcal{L}(H)/I(H)$ and recall a result of J. W. Calkin [1] that the natural involution on $\mathcal{L}(H)$ induces on $\mathcal{C}$ an involution so that $\mathcal{C}$ is a B^* -algebra. By the Gelfand-Neumark theorem we can consider $\mathcal{C}$ as an algebra of operators on a Hilbert space H_1 . If T is an operator, $\bar{T}$ denotes its image in $\mathcal{C}$ and we have the following relations:

1. T spectral implies $\bar{T}$ is spectral.
2. T nilpotent implies $\bar{T}$ is nilpotent.
3. $\bar{T}^* = \bar{T}^*$.
4. T hyponormal implies $\bar{T}$ is hyponormal.

A direct consequence of this fact is

LEMMA 3.1. *If T is a Riesz operator and hyponormal then T is compact.*

PROOF. Since T is a Riesz operator it is decomposable [9] and therefore $T = C + Q$. Then $\bar{T} = \bar{Q}$ and since $\bar{T}$ is hyponormal and quasinilpotent we have $\bar{T} = 0$. This completes the proof.

LEMMA 3.2. *If T is a spectral Riesz operator on Hilbert space and $(T^*T - TT^*)^k = C$ where C is compact, $k \geq 1$ then T is compact.*

PROOF. Since T is spectral $T = S + Q$ where S is its scalar part $S = \int_{\sigma(T)} \lambda dP(\lambda)$. It is known that [7], [3] in the above hypothesis T has its nilpotent part Q as a compact operator. Since for any arrangement of the λ_i 's, $S = \sum_{i=1}^{\infty} \lambda_i P_i$ is compact, we conclude that $\bar{T} = 0$ and hence T is compact.

LEMMA 3.3. *If T is a Riesz operator with the compact imaginary part then T is compact.*

PROOF. Let $T = A + iB$. It is clear that $\bar{T} = \bar{A}$ and since T is a Riesz operator it follows that T is nilpotent and therefore $\bar{T} = 0$.

Another proof follows from the fact that T is almost normal [7] which means that $T^*T - TT^* = C$, C is a compact operator and therefore $\bar{T}$ is normal and quasinilpotent.

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