ON EMBEDDING CONES OVER CIRCULARLY CHAINABLE CONTINUA

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It is known that every chainable continuum (i.e., every 1-cell-like continuum) can be embedded in the plane [1]. Every 2-cell-like continuum can be embedded in E^4 [5]. It is natural to ask whether every 2-cell-like continuum can be embedded in E^3 . Fearnley [3] has stated that the cone over a dyadic solenoid cannot be embedded in E^3 , but no proof has appeared. In this note we show that the cone over a nonplanar circularly chainable continuum cannot be embedded in E^3 . This provides a large class of 2-cell-like continua not embeddable in E^3 .

A definition of what it means to say that a continuum is P -like may be found in [6].

For any continuum M , the cone over M is the decomposition space $C(M)$ of the upper semicontinuous decomposition of $M \times [0, 1]$ whose only nondegenerate element is $M \times \{1\}$. We call the point $M \times \{1\}$ of $C(M)$ the vertex of $C(M)$ and the set of points $\{(m, 0), m \text{ in } M\}$, we call the base of $C(M)$.

It is easy to see that if P is a continuum and M is P -like then $C(M)$ is $C(P)$ -like. Therefore every cone over a circularly chainable continuum is 2-cell-like.

LEMMA. *Suppose M is a continuum and S is a closed subset of $C(M)$ which separates the vertex of $C(M)$ from the base of $C(M)$. Let v denote the vertex of $C(M)$, B the base of $C(M)$ and π the projection of $C(M) \setminus \{v\}$ onto B defined by $\pi(m, t) = (m, 0)$. There is a subcontinuum D of S such that $\pi(D) = B$.*

PROOF. We may assume that M is embedded in the Hilbert cube H and that $C(M)$ is embedded in $H \times [0, 1]$ as $\{(tp, t) \mid t \in [0, 1], p \in M\}$. Let $\{M_i\}_{i=1}^\infty$ be a sequence of locally connected subcontinua of H such that $M \subset M_i$ and M_i is contained in the $1/i$ neighborhood of M . Let S_i ($i = 1, 2, \dots$) be the closed $1/i$ neighborhood of S in $H \times [0, 1]$. For each i there is an integer $j(i)$ such that S_i separates $M_{j(i)}$ from v in $C(M_{j(i)}) (= \{(tp, t) \mid t \in [0, 1], p \in M_{j(i)}\})$. Since $C(M_{j(i)})$ is the cone over a locally connected continuum it is locally connected and unicoherent [2, Theorem A, p. 232] and thus some subcontinuum D_i of S_i separates $M_{j(i)}$ from v in $C(M_{j(i)})$ [7, Theorem 4.12, p. 51]. Let

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D be the limit of some convergent subsequence of the sequence $D_1, D_2, \dots$. Evidently D is a subcontinuum of S . If for some p in M , $\{(tp, t) \mid t \in [0, 1]\}$ misses D then this set misses D_i for some i , contradicting the fact that D_i separated v from $M_{j(i)}$ and hence v from M .

THEOREM. *If M is a nonplanar circularly chainable continuum then $C(M)$ cannot be embedded in E^3 .*

PROOF. Suppose h is an embedding of $C(M)$ into E^3 . Denote as B the base of $C(M)$ and as v the vertex of $C(M)$. There is a 2-sphere S in E^3 which separates $h(v)$ from $h(B)$. By the lemma there is a subcontinuum C of the intersection of S and $h(C(M))$ which can be mapped onto $h(B)$. But C could not be all of S since $h(B)$ is not locally connected. W. T. Ingram has shown [4] that there is no map of a planar continuum onto a nonplanar circularly chainable continuum, and we have a contradiction.

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