GROUPS DEFINED BY PERMUTATIONS OF A SINGLE WORD

WILLIAM EMERSON

In [1, p. 18] a technique of coset enumeration is used to show that the group on two generators r and s which satisfy the two relations

$$rs^2 = s^3r, \quad sr^2 = r^3s$$

is in fact the identity group. The question arises as to what relations in a presentation for a group force the group to "collapse" to the identity in general. This problem has been shown to be recursively unsolvable by Rabin [2], but in restricting our attention to a very special class of finitely presented groups generalizing the example above an answer is obtained.

DEFINITION. Let $W = W(x_1, \dots, x_n)$ be a word in the symbols $x_1, \dots, x_n$ (not all necessarily explicitly appearing). For $\sigma \in S_n$, the symmetric group on $\{1, 2, \dots, n\}$, σW denotes the word obtained from W by permuting the indices of the x_i according to $\sigma: x_i \rightarrow x_{\sigma(i)}$. Furthermore for any subset $\mathcal{O} \subseteq S_n$, we define the group

$$W^{\mathcal{O}} = \{x_1, \dots, x_n \mid \sigma W, \sigma \in \mathcal{O}\}.$$

In particular, if $W^{S_n} = \{1\}$ we say the word W is symmetrically trivial.

COMMENT. If W and V are freely equivalent we clearly have $W^{\mathcal{O}} \cong V^{\mathcal{O}}$, and hence we need only consider *reduced* words W . Secondly, a word W on n generators may also be formally considered to be on m generators for any $m > n$, in which case W^{S_m} and W^{S_n} are not generally isomorphic. In any case, the (common) order of the generators x_i in W^{S_m} divides their (common) order in W^{S_n} (since any relation in W^{S_n} is also in W^{S_m} for $m > n$), and hence if W is symmetrically trivial on n generators it is also symmetrically trivial when considered on $m > n$ generators.

The group W^{S_n} clearly has the property that any relation among $x_1, \dots, x_n$ remains valid after an arbitrary permutation of indices (in particular all x_i have the same order), and this principle is utilized in the proof of the next result which gives the explicit structure of the commutator quotient of W^{S_n} .

THEOREM 1. *W^{S_n} abelianized is isomorphic to*

$$(Z/dZ)^{n-2} \times (Z/\alpha Z) \times (Z/\beta Z)$$

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where we set $\alpha_i = \sigma_{z_i}(W)$ ($1 \leq i \leq n$), the exponent sum of x_i in W , and

$$d = \text{G.C.D.}\{\alpha_i - \alpha_j: 1 \leq i < j \leq n\} = \text{G.C.D.}\{\alpha_i - \alpha_1: 1 < i \leq n\}$$

$$\alpha = \text{G.C.D.}\{\alpha_i: 1 \leq i \leq n\} = \text{G.C.D.}\{\alpha_1, d\}$$

$$\beta = \frac{d}{\alpha} \left| \sum_{1 \leq i \leq n} \alpha_i \right|.$$

We use the conventions $\text{G.C.D.}\{0\} = 0$ and $Z/0Z = Z$.

COROLLARY. W^{S_n} abelianized is trivial iff

$$d = 1 \quad \text{and} \quad \sum_{1 \leq i \leq n} \alpha_i = \pm 1,$$

and hence these conditions are necessary for W to be symmetrically trivial.

PROOF. The corollary is immediate since $d=1 \Rightarrow \alpha=1$. Also, the second equalities in the definitions of d and α follow from the first since $(\alpha_i - \alpha_1) - (\alpha_j - \alpha_1) = \alpha_i - \alpha_j$, and we could just have readily used any "fixed" subscript other than 1.

Let $W(x_1, \dots, x_n) = x_{a_1}^{e_1} x_{a_2}^{e_2} \cdots x_{a_k}^{e_k}$. Then W^{S_n} abelianized

$$\begin{aligned} &= \{x_1, \dots, x_n \mid x_i x_j x_i^{-1} x_j^{-1} (1 \leq i, j \leq n), \sigma W (\sigma \in S_n)\} \\ (1) \quad &= \left\{x_1, \dots, x_n \mid x_i x_j x_i^{-1} x_j^{-1} (1 \leq i, j \leq m), \sigma \left(\prod_i x_i^{\alpha_i} \right) (\sigma \in S_n) \right\}, \end{aligned}$$

upon "abelianizing" W and recalling the definition of the α_i . We now consider the transpositions $\sigma_{i,j} \in S_n$ interchanging i and j , and note that modulo commutators

$$(2) \quad \sigma_{i,j} \left(\prod_i x_i^{\alpha_i} \right) = \left(\prod_i x_i^{\alpha_i} \right) (x_i^{\alpha_j - \alpha_i} x_j^{\alpha_i - \alpha_j}).$$

But any permutation σ is the product of transpositions, and consequently modulo commutators we have for any $\sigma \in S_n$: $\sigma \left(\prod_i x_i^{\alpha_i} \right) = \left(\prod_i x_i^{\alpha_i} \right) T_1 \cdots T_k$ where each T_i is of the form $(x_r^{\alpha_j - \alpha_i} x_s^{\alpha_i - \alpha_j})$ for some choice of $i, j, r, s \in \{1, \dots, n\}$. Each such T_i is a relator in W^{S_n} abelianized since $x_i^{\alpha_j - \alpha_i} x_j^{\alpha_i - \alpha_j}$ is by (2) and we may freely permute indices. Now it is easy to see that this collection of relators in W^{S_n} abelianized is equivalent to the relations

$$(3) \quad x_i^d = x_j^d \quad (1 \leq i < j \leq n) \quad d = \text{G.C.D.}\{\alpha_i - \alpha_j: 1 \leq i < j \leq n\}.$$

Furthermore, $\prod_i x_i^{\alpha_i} = (x_1 \cdots x_n)^{\alpha_1} \prod_i x_i^{\alpha_i - \alpha_1}$ implying by (3) that in W^{S_n} abelianized

$$(4) \quad \prod_i x_i^{\alpha_i} = (x_1 \cdots x_n)^{\alpha_1} x_1^{-D} \text{ where } D = \sum_i (\alpha_1 - \alpha_i).$$

Therefore W^{S_n} abelianized

$$= \{x_1, \dots, x_n \mid x_i x_j x_i^{-1} x_j^{-1} \ (1 \leq i, j \leq n), \\ (x_1 \cdots x_n)^{\alpha_1} x_1^{-D}, x_i x_j^{-d} \ (1 \leq i < j \leq n)\},$$

that is a set of defining relations for W^{S_n} abelianized (in addition to the abelian condition) is

$$(5) \text{ (i) } x_i^d = x_j^d, \ 1 \leq i < j \leq n,$$

$$\text{(ii) } (x_1 \cdots x_n)^{\alpha_1} = x_1^D.$$

Now define $\alpha = \text{G.C.D. } \{\alpha_1, d\} = \text{G.C.D. } \{\alpha_i: 1 \leq i \leq n\}$, and let $\alpha = r\alpha_1 + sd$, $r, s \in \mathbb{Z}$. From (i) we obtain

$$\text{(iii) } (x_1 \cdots x_n)^d = x_1^{nd}, \text{ and from (ii) and (iii),}$$

$$(6) \text{ (iv) } (x_1 \cdots x_n)^\alpha = x_1^F \text{ where } F = rD + snd,$$

and $d \mid F$ since $d \mid D$. Since $\alpha \mid \alpha_1$ (iv) implies $(x_1 \cdots x_n)^{\alpha_1} = x_1^{F\alpha_1/\alpha}$, and thus from (ii) $x_1^D = x_1^{F\alpha_1/\alpha}$. Hence we may replace (5) by

$$(7) \text{ (i) } x_i^d = x_1^d, \ 1 < i \leq n,$$

$$\text{(ii)' } (x_1 \cdots x_n)^\alpha = x_1^F,$$

$$\text{(iii)' } x_1^D = x_1^{F\alpha_1/\alpha}.$$

Now consider these relations with respect to the new generators

$$(8) \quad y_i = x_1 \cdots x_i x_1^{-i}, \quad 1 \leq i < n, \\ y_n = x_1 \cdots x_n x_1^{-F/\alpha}.$$

They become

$$\text{(i) } y_i^d = 1, \ 1 < i < n, \quad y_1^{(Fd/\alpha) - nd} = 1,$$

$$\text{(ii) } y_n^\alpha = 1,$$

$$\text{(iii) } y_1^{(F\alpha_1/\alpha) - D} = 1.$$

Hence we see that W^{S_n} abelianized is the direct product of the cyclic groups generated by $y_1, \dots, y_n$, and all that remains is the explicit computation of β , the order of y_1 . From (i) and (iii) we have

$$\beta = \text{G.C.D.} \{F\alpha_1/\alpha - D, Fd/\alpha - nd\},$$

and

$$\frac{\alpha_1}{\alpha} \left(F \frac{d}{\alpha} - nd \right) - \frac{d}{\alpha} \left(F \frac{\alpha_1}{\alpha} - D \right) = \frac{d}{\alpha} (D - n\alpha_1) \\ \Rightarrow \beta \mid \frac{d}{\alpha} \sum_i \alpha_i, \text{ by (4).}$$

on the other hand, we conclude by verifying that

$$\frac{d}{\alpha}(n\alpha_1 - D) = \frac{d}{\alpha} \sum_i \alpha_i$$

divides both $(Fd/\alpha) - nd$ and $(F\alpha_1/\alpha) - D \Rightarrow \beta = (d/\alpha) \mid \sum_i \alpha_i \mid$. From (6), $\alpha = r\alpha_1 + sd$, $F = rD + snd$ implying

$$\begin{aligned} (Fd/\alpha) - nd &= (d/\alpha)rD + (d/\alpha)snd - nd \\ &= (d/\alpha)rD + (d/\alpha)n(\alpha - r\alpha_1) - nd = -r((d/\alpha)(n\alpha_1 - D)); \\ (F\alpha_1/\alpha) - D &= (\alpha_1/\alpha)rD + (\alpha_1/\alpha)snd - D \\ &= (D/\alpha)(\alpha - sd) + (\alpha_1/\alpha)snd - D = s((d/\alpha)(n\alpha_1 - D)). \end{aligned}$$

We now consider words W in two generators, in which case a necessary condition for the symmetric triviality of W is $\{|\alpha_1|, |\alpha_2|\} = \{0, 1\}$ by the corollary. It is too much to hope that this condition is also necessary in general since it only depends on the exponential sums of the generators in W . Nevertheless, we have the following result which generalizes the example of [1]:

THEOREM 2. *If $W(x, y)$ is any word in x and y of syllable length less than six, W is symmetrically trivial $\Leftrightarrow \{|\alpha_1|, |\alpha_2|\} = \{0, 1\}$.*

PROOF. Without loss of generality we may assume W is cyclically reduced, and also (interchanging x and y and/or replacing W by W^{-1} if necessary) that

$$W(x, y) = x^{-(a+\epsilon)}y^bx^ay^{-b}$$

where $\epsilon = +1$ or -1 , $a \in N$ and $b \in Z$. Hence the two basic defining relations are

$$(I) \quad y^bx^ay^{-b} = x^{a+\epsilon},$$

$$(II) \quad x^by^ax^{-b} = y^{a+\epsilon}.$$

Relation (II) implies

$$(9) \quad x^by^{ab}x^{-b} = y^{(a+\epsilon)b},$$

while (I) implies

$$y^bx^{a^2}y^{-b} = x^{(a+\epsilon)a}, \quad y^{2b}x^{a^2}y^{-2b} = x^{(a+\epsilon)^2},$$

and by induction on n

$$(10) \quad y^{nb}x^{a^n}y^{-nb} = x^{(a+\epsilon)^n} \quad (n \in N).$$

Setting $n = a + \epsilon$ in (10) we obtain

$$y^{(a+\epsilon)b} x^{a(a+\epsilon)} = x^{(a+\epsilon)^{a+\epsilon}} y^{(a+\epsilon)b},$$

which with (9) implies

$$(x^b y^{ab} x^{-b}) x^{a(a+\epsilon)} = x^{(a+\epsilon)^{a+\epsilon}} (x^b y^{ab} x^{-b}) \Rightarrow y^{ab} x^{a(a+\epsilon)} = x^{(a+\epsilon)^{a+\epsilon}} y^{ab}$$

or

$$(11) \quad y^{ab} x^{a(a+\epsilon)} y^{-ab} = x^{(a+\epsilon)^{a+\epsilon}}.$$

Setting $n = a$ in (10) we obtain

$$(12) \quad y^{ab} x^{a^2} y^{-ab} = x^{(a+\epsilon)^a}.$$

We consider two cases:

If $\epsilon = +1$ we obtain from (12)

$$(12)' \quad y^{ab} x^{a(a+\epsilon)} y^{-ab} = x^{a(a+\epsilon)^a},$$

while if $\epsilon = -1$ we obtain from (11)

$$(11)' \quad y^{ab} x^{a^2} y^{-ab} = x^{a(a+\epsilon)^{a+\epsilon}}.$$

Consequently, if $\epsilon = +1$ (11) and (12)' $\Rightarrow$

$$(13) \quad x^{(a+\epsilon)^a} = 1,$$

and if $\epsilon = -1$ (11)' and (12) $\Rightarrow x^{(a+\epsilon)^a} = x^{a(a+\epsilon)^{a+\epsilon}}, x^{(a+\epsilon)^{a+\epsilon}} = 1, \Rightarrow$

$$(13)' \quad x^{(a+\epsilon)^a} = 1.$$

In either case, using (13) in conjunction with (12) we obtain

$$(14) \quad x^{a^2} = 1,$$

which, since $(a^a, (a+\epsilon)^{a+\epsilon}) = 1$, implies $x = 1$, and hence by symmetry $y = 1$ also.

REFERENCES

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COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY