A THEOREM ON MULTIPLIERS OF TYPE (p, q)

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In [1], we established that the convolutions of functions in L^p and in L^q , $1/p + 1/q = 1$, are r -multipliers for an appropriate r . In this short note, we investigate the case that p and q are no longer conjugate indices.

Let G and $\hat{G}$ be two locally compact abelian groups, dual in the sense of Pontrjagin, and $L^p(G)$ be the Banach space of equivalence classes of p th power Haar integrable functions with the usual norm. Let $F: L^2(G) \rightarrow L^2(\hat{G})$ and $\bar{F}: L^2(\hat{G}) \rightarrow L^2(G)$ be Fourier transform mappings. Haar measures are chosen so that F and $\bar{F}$ are isometries.

A measurable function f on $\hat{G}$ is called a multiplier of type (p, q) , if

$$\sup\{\|\bar{F}fF\phi\|_q/\|\phi\|_p; \phi \in S(G), \phi \neq 0\}$$

is finite, where $S(G)$ is the space of simple functions on G . The space of all multipliers of type (p, q) will be denoted by $M_p^q = M_p^q(\hat{G})$. Then we have (see [1])

LEMMA. Let $1/p + 1/p' = 1/q + 1/q' = 1$. The following are equivalent:

- (i) $f \in M_p^q(\hat{G})$.
- (ii) There is a constant K such that

$$\left| \int_{\hat{G}} f \cdot F\phi \cdot F\psi \right| \leq K \|\phi\|_p \cdot \|\psi\|_{q'}$$

for all ϕ and ψ in $S(G)$.

Now define a multilinear mapping

$$T: S(\hat{G}) \times S(\hat{G}) \times S(G) \times S(G) \rightarrow \mathbf{C},$$

where $\mathbf{C}$ is the complex field, by

$$T(f, g, \phi, \psi) = \int_{\hat{G}} (f * g)(\phi * \psi)^{\wedge}.$$

Then in [1] (see proof of Theorem 5 there), we showed that

$$(1) \quad |T(f, g, \phi, \psi)| \leq \|f\|_{p_1} \cdot \|g\|_{q_1} \cdot \|\phi\|_{r_1} \cdot \|\psi\|_{s_1},$$

with $1/p_1 + 1/q_1 = 1$, $1/r_1 + 1/s_1 = 1$, $1/p_1 + 1/r_1 = 3/2$, $p_1 \leq q_1$.

On the other hand,

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$$\begin{aligned}
|T(f, g, \phi, \psi)| &= \left| \int_{\hat{G}} (f * g)(\phi * \psi)^{\wedge} \right| \\
&\leq \|f * g\|_t \cdot \|(\phi * \psi)^{\wedge}\|_{t'} \quad \left(\frac{1}{t} + \frac{1}{t'} = 1, 1 \leq t \leq 2 \right) \\
&\leq \|f\|_{p_2} \cdot \|g\|_{q_2} \cdot \|\phi * \psi\|_t \quad (1/p_2 + 1/q_2 - 1 = 1/t). \\
\therefore |T(f, g, \phi, \psi)| &\leq \|f\|_{p_2} \cdot \|g\|_{q_2} \cdot \|\phi\|_{r_2} \cdot \|\psi\|_{s_2}
\end{aligned}$$

for $1/2 \leq 1/p_2 + 1/q_2 - 1 = 1/r_2 + 1/s_2 - 1 \leq 1$.

Therefore by the multilinear version of the Riesz-Thorin interpolation theorem (Zygmund [3, XII, Theorem 3.3]),

$$|T(f, g, \phi, \psi)| \leq \|f\|_p \cdot \|g\|_q \cdot \|\phi\|_r \cdot \|\psi\|_s$$

holds for $1/p = (1-\lambda)/p_1 + \lambda/p_2$, $1/q = (1-\lambda)/q_1 + \lambda/q_2$,

$1/r = (1-\lambda)/r_1 + \lambda/r_2$, $1/s = (1-\lambda)/s_1 + \lambda/s_2$ ($0 \leq \lambda \leq 1$).

Or,

$$\begin{aligned}
1/p + 1/q - 1 &= \lambda(1/p_2 + 1/q_2 - 1), \\
1/r + 1/s - 1 &= \lambda(1/r_2 + 1/s_2 - 1), \\
1/p + 1/r - 3/2 &= \lambda(1/p_2 + 1/r_2 - 3/2) \quad (0 \leq \lambda \leq 1).
\end{aligned}$$

In particular, if we put $p_2 = q_2 = r_2 = s_2 = 1$, and solve r and s in terms of p and q , we obtain

$$1/r = 1 - 1/(2p) + 1/(2q), \quad 1/s = 3/(2p) + 1/(2q) - 1.$$

Hence by the Lemma, we get

$$L^p * L^q \subset M_r^{s'} \quad (1/s + 1/s' = 1, \text{ or } 1/s' = 2 - 3/(2p) - 1/(2q)).$$

Now applying Theorem 1.3 in Hörmander [2], we have

THEOREM. If $1 \leq p \leq q$, $1/p + 1/q \geq 1$, then $L^p(\hat{G}) * L^q(\hat{G}) \subset M_r^s(\hat{G})$ where

$$\begin{aligned}
1/r &= (1 - 2\theta) + (4\theta - 1)/(2p) + 1/(2q), \\
1/s &= 2(1 - \theta) + (4\theta - 3)/(2p) - 1/(2q),
\end{aligned}$$

for all θ , $0 \leq \theta \leq 1$.

From this theorem, one can obtain extensions of most of the results in [1]; as the reasoning goes parallel to that employed there, we leave it for the interested readers.

OPEN QUESTION. If $1/p + 1/q = 1$, our result agrees with that of [1]. Thus it cannot be improved in this case. It will be interesting to know whether our bounds are correct in general.

REFERENCES

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