ANALYTIC CONTINUATION OF THE S-MATRIX FOR POTENTIAL SCATTERING¹

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In this note, we propose to give shortened proofs of two theorems of Dolph, McLeod and Thoe [4] on the analytic continuation of the resolvent kernel and S -matrix for potential scattering to the "unphysical sheet." The proofs will use Carleman's inequality and the currently fashionable method of factoring the potential. (See [5], [9] and the references given there.) Since our purpose is to illustrate a method, we shall consider the problem in dimension three only, although McLeod [10] has extended the results of [4] to arbitrary dimensions.

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R_3 will denote real Euclidean three-space and Δ the Laplacian operator on $L_2(R_3)$. Integrals without an indicated domain of integration are over all of R_3 .

We shall adopt the hypotheses of [4].

ASSUMPTION. Let $V(x)$ be a real-valued function on R_3 such that

(1) $V(x)$ is locally Holder continuous except for a finite number of square integrable singularities, and

(2) there exist positive constants C, α and R such that

$$|V(x)| \leq Ce^{-\alpha|x|} \quad \text{for } |x| \geq R.$$

The Schroedinger operator $T = -\Delta + V(x)$ is then selfadjoint with domain $D(T) = D(\Delta)$ [8].

LEMMA 1. If $f, g \in L_2(R_3)$, then $\iint |f(x)g(y)| |x-y|^{-2} dy dx < +\infty$.

This is an immediate consequence of Sobolev's inequality [1, p. 220, part (c)].

The following fact is well known, but, having no convenient reference for it, we supply a brief proof.

LEMMA 2. Let (μ, X) be a measure space and $f(\cdot, k) \in L_2(\mu, X)$ for each k in a domain Ω of the complex plane. Assume that for a.e. x in X , $f(x, k)$ is analytic on Ω with k -derivative $f'(x, k)$. If $|f(x, k)| \leq g(x)$ for some $g \in L_2(\mu, X)$, then $f(\cdot, k)$ is a vector-valued analytic function on Ω and its derivative is $f'(\cdot, k)$.

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PROOF. If k is fixed and Ω contains $\{z: |z-k| \leq 2r_0\}$, then for $|h| \leq r_0$

$$f(x, k+h) - f(x, k) = (h/2\pi i) \int_{|z-k|=2r_0} (z-k)^{-1}(z-k-h)^{-1}f(x, z)dz.$$

A simple estimate shows that $h^{-1}[f(\cdot, k+h) - f(\cdot, k)]$ is majorized by $(2/r_0)g(x)$ for $|h| \leq r_0$, and hence converges to $f'(\cdot, k)$ in the norm of $L_2(\mu, X)$.

LEMMA 3. Let $Q(k)$ be an operator-valued function analytic in Hilbert-Schmidt norm on a domain Ω . If $I+Q(k)$ is invertible for some k in Ω , then $[I+Q(k)]^{-1}$ exists except at isolated points and is meromorphic on Ω .

PROOF. The points where $I+Q(k)$ is not invertible are the zeros of $\det_2[I+Q(k)]$ [3, p. 1106]. If P_n is a sequence of finite rank orthogonal projections increasing strongly to I , then $P_n Q(k) P_n \rightarrow Q(k)$ in Schmidt norm for each k in Ω . The argument in the proof of part (c) of Lemma XI. 9.22 of [3] (pp. 1109–1110) shows that $\det_2[I+Q(k)]$ is analytic, so that $I+Q(k)$ is invertible except at isolated points. These points are poles of finite order by virtue of the Carleman inequality [3, Corollary 25, p. 1112]

$$\|(\det_2[I+Q(k)])[I+Q(k)]^{-1}\| \leq \exp(C_2\|Q(k)\|_2^2)$$

where C_2 is a constant.

Let $R_0(z) = (-\Delta - z)^{-1}$. If $\text{Im } k > 0$ and k^2 is in the resolvent set of $-\Delta$, then $R_0(k^2)$ is an integral operator with kernel

$$G_0(x, y, k) = (1/4\pi) |x - y|^{-1} e^{ik|x-y|}.$$

Define $A(x) = e^{-\alpha|x|/2}$ and $B(x) = e^{\alpha|x|/2}V(x)$, and let A and B denote the corresponding multiplication operators. Note that $V(x) = A(x)B(x)$.

THEOREM 1. (a) For $\text{Im } k > 0$, $AR_0(k^2)B$ has a unique extension to a Hilbert-Schmidt operator $Q(k)$ on $L_2(R_3)$. $Q(k)$ is analytic for $\text{Im } k > 0$ and has an analytic continuation to $\text{Im } k > -\alpha/2$ in Hilbert-Schmidt norm.

(b) $I+Q(k)$ has a bounded inverse for $\text{Im } k > -\alpha/2$, except for isolated values of k . $[I+Q(k)]^{-1}$ is meromorphic on $\text{Im } k > -\alpha/2$, and the poles of $[I+Q(k)]^{-1}$ in $\text{Im } k > 0$ are pure imaginary, finite in number and occur when k^2 is a negative eigenvalue of T .

PROOF. For $\text{Im } k > 0$, $AR_0(k^2)B$ has the kernel

$$(1) \quad K(x, y, k) = (1/4\pi) |x - y|^{-1} A(x) e^{ik|x-y|} B(y).$$

If $y \neq x$, (1) defines $K(x, y, k)$ as an entire function of k , which is majorized for $\text{Im } k \geq -\eta_0 > -\alpha/2$ by

$$K_0(x, y) = (1/4\pi) |x - y|^{-1} |A(x)| e^{\eta_0|x-y|} |B(y)|.$$

But $\iint |K_0(x, y)|^2 dx dy$ does not exceed

$$(1/16\pi^2) \int_{|x| \geq R} \int_{|y| \geq R} |x - y|^{-2} e^{-(\alpha-2\eta_0)|x|} e^{-(\alpha-2\eta_0)|y|} dx dy \\ + C_1 \int_{|x| \leq R} \int_{|y| \leq R} |x - y|^{-2} |A(x)| |B(y)| dx dy$$

where C_1 is a constant. If these integrals are trivially extended to the whole space, both are seen to be convergent by Lemma 1. Thus for $\text{Im } k > -\alpha/2$, $K(x, y, k)$ is the kernel of a Hilbert-Schmidt operator, which agrees with $AR_0(k^2)B$ for $\text{Im } k > 0$. The analyticity statements follow from the uniform majorization by $K_0(x, y)$ and Lemma 2.

If $\text{Im } k > 0$ and k^2 is not real, then $I + Q(k)$ has the bounded inverse $I - A(T - k^2)^{-1}B$ [9]. It follows by Lemma 3 that $[I + Q(k)]^{-1}$ is meromorphic for $\text{Im } k > -\alpha/2$. Finiteness of the number of poles in the upper half-plane is proved in [9].

In the present case, the S -matrix $S(k)$ is defined for $k > 0$ and is a unitary operator on $L_2(\Sigma)$ where Σ is the unit sphere in R_3 with surface measure. According to Ikebe [7, Theorem 1], $S(k) - I$ is a Hilbert-Schmidt operator on $L_2(\Sigma)$ with kernel

$$t(\omega, \omega', k) = (-ik/8\pi^2) \int \phi(x, \omega, k) V(x) e^{ik\omega' \cdot x} dx$$

where $\omega, \omega' \in \Sigma$. $\phi(x, \omega, k)$ is bounded and continuous on R_3 and satisfies

$$(2) \quad \phi(x, \omega, k) = e^{-ik\omega \cdot x} - \int G_0(x, y, k) V(y) \phi(y, \omega, k) dy.$$

THEOREM 2. $S(k)$ has a meromorphic continuation to $|\text{Im } k| < \alpha/2$, which has poles only at the poles of $[I + Q(k)]^{-1}$.

PROOF. Let $a(x, \omega, k) = A(x)e^{-ik\omega \cdot x}$ and $b(x, \omega, k) = B(x)e^{-ik\omega \cdot x}$. Multiplying (2) by $A(x)$ and observing that a, b and $A\phi$ are in $L_2(R_3)$, we find after a simple calculation that

$$(3) \quad t(\omega, \omega', k) = (b(\omega, k), [I + Q(k)]^{-1}a(\omega', k))_{L_2(R_3)}.$$

Now for each $\omega \in \Sigma$, $a(\omega, k)$ has an analytic continuation to $\text{Im } k > -\alpha/2$ which is bounded in $L_2(R_3)$ -norm, uniformly on

$\Sigma \times \{k: \operatorname{Im} k \geq -\eta_0\}$ for any $\eta_0 < \alpha/2$. $b(\omega, k)$ has a similar *conjugate analytic* continuation to $|\operatorname{Im} k| \leq \eta_0$, which is bounded uniformly on $\Sigma \times \Sigma$ by $\| [I + Q(k)]^{-1} \|$ times a constant depending on η_0 . Analyticity in Hilbert-Schmidt norm of the kernel operator $S(k) - I$, except at poles of $[I + Q(k)]^{-1}$ now follows from Lemma 2.

If $\operatorname{Im} k > 0$ and k^2 is in the resolvent set of T , then [6, Theorem 1] $(T - k^2)^{-1}$ has a kernel $G(x, y, k)$ of Carleman type, which satisfies

$$(4) \quad G(x, y, k) = G_0(x, y, k) - \int G_0(x, \xi, k) V(\xi) G(\xi, y, k) d\xi$$

for a.e. fixed y . Multiplying (4) by $A(x)$, we find that the function $g(x, y, k) = A(x)G(x, y, k)$ satisfies

$$(5) \quad g(\cdot, y, k) = [I + Q(k)]^{-1} g_0(\cdot, y, k)$$

for a.e. fixed y , where $g_0(x, y, k) = A(x)G_0(x, y, k)$. Since $g_0(\cdot, y, k)$ has an obvious continuation to $\operatorname{Im} k > -\alpha/2$ as an element of $L_2(R_2)$, (5) gives a meromorphic continuation of $g(\cdot, y, k)$. Recalling that $A(x) = e^{-\alpha|x|/2}$ yields

THEOREM 3. *Let $\mathcal{H} = L_2(\mu, R_3)$ where $d\mu = e^{-\alpha|x|} dx$. Then $G(\cdot, y, k)$ has an $\mathcal{H}$ -valued meromorphic continuation to $\operatorname{Im} k > -\alpha/2$, which has poles only at the poles of $[I + Q(k)]^{-1}$.*

REMARKS. (1) In Theorems 2 and 3, we assert only that any pole of $S(k)$ or $G(\cdot, y, k)$ must also be a pole of $[I + Q(k)]^{-1}$; the converse is not established.

(2) Lemma 3 is by no means the best such result. A generalization to $Q(k)$ compact on a Hilbert space follows from [3, VII 6.13, p. 592] and [2, Appendix, Theorem 18]. Moreover, the referee has kindly informed us that the lemma has been proved for $Q(k)$ compact on a Banach space by Stanley Steinberg of Purdue and, for a separable Banach space with basis, by Peter Werner and H. Haf of Technische Hochschule, Stuttgart. These papers should appear in print soon, and are presently available as preprints.

(3) Finally, we remark that the method employed here should be generally useful in studying S -matrix analyticity, provided that (a) the operator $AR_0(z)B$ is compact and can be continued across the real axis, and (b) one has a suitable representation of the S -matrix in terms of the boundary values of this operator.

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