SUBFUNCTION STRUCTURE

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Of prime interest to us here is Theorem 4 in the proof of which an interesting structure is revealed by (2), (3), and (4). An easy consequence of Theorem 4 is Theorem 5 which embraces results of specific interest. One such result is Theorem B.4 stated and proved in the Amer. Math. Monthly **74** (1967), 773–776. Another such result¹ is a Theorem of Erdős and Szekeres² succinctly proved by A. Seidenberg in the J. London Math. Soc. **34** (1959), 352. In both these proofs a central trick appears and then reappears here in Theorem 4 (1) below.

We assume that a relation is a set of ordered pairs and, in the usual way, that a function is a special sort of relation.

We agree that $\mathfrak{R}$ is the set of all real numbers, that $\mathfrak{F}$ is the set of all those functions whose domains are subsets of $\mathfrak{R}$, that ω is the set of integers n for which $n \geq 0$, that $\sim S$ is the complement of S with respect to the universe, and that

$$\text{pwr}'S = \sum_{x \in S} 1.$$

Thus $\text{pwr}'S$ = the number of points in S .

We agree that x and y are *consecutive* in S if and only if x and y are such members of S that $x < y$ and there is no $z \in S$ for which

$$x < z < y.$$

With a relation R tentatively in mind we agree that $\text{up}R$ is the set of $f \in \mathfrak{F}$ for which

$$(f(x), f(y)) \in R$$

whenever x and y are consecutive in the domain of f .

We also agree that $\text{rise}R$ is the set of $f \in \mathfrak{F}$ for which

$$(f(x), f(y)) \in R$$

whenever x and y are such members of the domain of f that $x < y$.

1. THEOREM. *If f is on S then $\text{pwr}'f = \text{pwr}'S$.*
2. THEOREM. $\text{rise}R \subset \text{up}R$.
3. THEOREM. *If R is transitive, $\text{pwr}'f \in \omega$, $f \in \text{up}R$, then $f \in \text{rise}R$.*

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¹ P. L. Montgomery has shown me that this result follows from the first.

² *Compositio Math.* **2** (1935), 467–468.

Applications of Theorems 4 and 5 arise by taking R to be some appropriate relation. For example we might so choose R that $(s, t) \in R$ if and only if $s < t$. Again we might so choose R that $(s, t) \in R$ if and only if $1 \leq s \in \omega$ and $2 \leq t/s \in \omega$. Still again we might so choose R intransitively that $(s, t) \in R$ if and only if $s \subset t \subset \aleph$ or $t \subset s \subset \aleph$.

4. THEOREM. If $f \in \mathfrak{F}$, $\text{pwr}'f \in \omega$, $\mathfrak{S} =$ the set of $h \subset f$ for which $h \in \text{up } R$, $\mathfrak{R} =$ the set of $h \subset f$ for which $h \in \text{rise} \sim R$, $\mu = \sup_{h \in \mathfrak{S}} \text{pwr}'h$, $\nu = \sup_{h \in \mathfrak{R}} \text{pwr}'h$, then

$$\mu \in \omega, \quad \nu \in \omega, \quad \text{pwr}'f \leq \mu \cdot \nu.$$

PROOF. Let $D =$ the domain of f . For $x \in D$, let $H_x =$ the set of $h \in \mathfrak{S}$ for which $x =$ the supremum of the domain of h . For $x \in D$, let

$$(1) \quad N_x = \sup_{h \in H_x} \text{pwr}'h.$$

Let

$$(2) \quad P = \text{the set of } p \in \omega \quad \text{for which } 1 \leq p \leq \mu.$$

For $p \in P$ let

$$A_p = \text{the set of } x \in D \quad \text{for which } N_x = p.$$

For $p \in P$ let

$$\psi_p = \text{the restriction of } f \text{ to } A_p.$$

Clearly the empty function belongs to $\mathfrak{S} \cap \mathfrak{R}$. Also

$$\begin{aligned} \mu \in \omega, \quad \nu \in \omega, \\ N_x \in P \quad \text{whenever } x \in D, \\ D = \bigcup_{p \in P} A_p, \end{aligned}$$

and hence

$$(3) \quad f = \bigcup_{p \in P} \psi_p.$$

Now if

$$p \in P, \quad x \in A_p, \quad y \in A_p, \quad x < y, \quad (f(x), f(y)) \in R,$$

then absurdly

$$p = N_x < N_x + 1 \leq N_y = p.$$

Consequently, and this is crucial,

(4) $\psi_p \in \mathfrak{R}$ whenever $p \in P$.

Because of (3), (4), the nature of ν , and (2)

$$\begin{aligned} \text{pwr}'f &\leq \sum_{p \in P} \text{pwr}'\psi_p \leq \sum_{p \in P} \nu \\ &= \nu \cdot \text{pwr}'P = \nu \cdot \mu = \mu \cdot \nu. \end{aligned}$$

An easy consequence of Theorem 4 is

5. THEOREM. *If*

$$f \in \mathfrak{F}, \quad m \in \omega, \quad n \in \omega, \quad \text{pwr}'f = 1 + m \cdot n,$$

then there is a $g \subset f$ for which either

$$\text{pwr}'g = 1 + m \quad \text{and} \quad g \in \text{up } R$$

or

$$\text{pwr}'g = 1 + n \quad \text{and} \quad g \in \text{rise} \sim R.$$

An easy consequence of Theorem 5 is

6. THEOREM. *If*

$$\begin{aligned} S \subset \mathfrak{R}, \quad f \text{ is on } S \text{ to } \mathfrak{R}, \quad m \in \omega, \quad n \in \omega, \\ \text{pwr}'f = 1 + m \cdot n, \end{aligned}$$

then there is a $g \subset f$ for which either

$$\text{pwr}'g = 1 + m \quad \text{and} \quad g \text{ is increasing}$$

or

$$\text{pwr}'g = 1 + n \quad \text{and} \quad g \text{ is nonincreasing.}$$

Even here $\text{pwr}'f$ is sharply estimated.

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