THE AREA OF AN ANALYTIC SET IN COMPLEX PROJECTIVE SPACE

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This note consists of a proof of the fact that the area of an analytic set in complex projective space is equal to the product of the degree of the set times a constant, the constant being dependent only on the dimension of the analytic set. A proof of this theorem is sketched by de Rham in [1]. The proof given here is quite distinct, and is a modification of Proposition 2.4 of [2].

Let W be an $(n+1)$ -dimensional complex vector space, $P(W)$ the associated projective space, and A a q -dimensional analytic set in $P(W)$. Let ω_q be the $2q$ -dimensional volume element in $P(W)$. For a nonnegative integer p , let $c(p) = p!/\pi^p$.

THEOREM. *Let $\deg A$ denote the degree of A . Then*

$$c(q) \int_A \omega_q = \deg A.$$

PROOF. By induction on $d = n - q$. Clearly it is sufficient to prove the theorem for A irreducible. Now if $d = 0$, $A = P(W)$ and the theorem is trivial. Assume the theorem is true for $n - q \leq d - 1$. Let A be a q -dimensional analytic set in $P(W)$, where W is an $(n+1)$ -dimensional complex vector space, and where $n - q = d \geq 1$. If $n = 1$, $q = 0$ and the result follows. Thus assume $n \geq 2$.

Let G be the Grassman manifold of $(n - q)$ -dimensional planes in $P(W)$. For an arbitrary set S , let $\#S$ denote the number of elements of S , setting $\#S = 0$ if $S = \emptyset$ and $\#S = \infty$ if S is infinite. Then $L \in G$ and $\#\{L \cap A\} < \infty$ implies that $\#\{L \cap A\} \leq \deg A$. Moreover, there exists a thin analytic set T in G such that $L \in G - T$ implies that $\#\{L \cap A\} = \deg A$. Take any $L \in G - T$. Take any point $a \in L - A$. Choose an orthonormal basis of W in such a way that if W is identified with $\mathbb{C}^{n+1}$ and $P(W)$ with $P(\mathbb{C}^{n+1}) = P^n$, and if $\rho: \mathbb{C}^{n+1} - \{0\} \rightarrow P^n$ is the residual map, then $b = (1, 0, \dots, 0) \in \mathbb{C}^{n+1}$ is in $\rho^{-1}(a)$, and $\rho^{-1}(L) = \{(z_0, \dots, z_{n-q}, 0, \dots, 0) \mid (z_0, \dots, z_{n-q}) \neq 0\}$. Let $P^{n-1} = P(\mathbb{C}^n)$ and $\tau: \mathbb{C}^n - \{0\} \rightarrow P^{n-1}$ the residual map. Define $\pi: P^n - \{a\} \rightarrow P^{n-1}$ by $\pi(\rho(z_0, \dots, z_n)) = \tau(z_1, \dots, z_n)$. Let $\pi(A) = B$. Proposition 2.4 of [2] yields the following result: "The map $\pi|A: A \rightarrow B$ is proper and light, B is an irreducible q -dimensional analytic set in P^{n-1} and there exists an analytic set $S \subset B$, thin in B , such that $\pi|A - \pi^{-1}(S): A$

Received by the editors March 28, 1968.

$-\pi^{-1}(S) \rightarrow B-S$ is a regular covering map of the complex manifold $A-\pi^{-1}(S)$ onto the connected manifold $B-S$, with covering number λ . Moreover, if ω'_q is the volume element of P^{n-1} , then $c(q)\int_A \omega_q = \lambda c(q)\int_B \omega'_q$ holds."

Now let $\pi(L-\{a\})=L'$. Then L' is an $(n-q-1)$ -dimensional plane in P^{n-1} . Since $\#\{A \cap L\} < \infty$, $\#\{B \cap L'\} < \infty$. Using the induction hypothesis, we have $\#\{B \cap L'\} \leq \deg B = c(q)\int_B \omega'_q$. But for any $x \in B$, $\#\{\pi^{-1}(x) \cap A\} \leq \lambda$. Therefore $\deg A = \#\{A \cap L\} \leq \lambda(\#\{B \cap L'\}) \leq \lambda c(q)\int_B \omega'_q = c(q)\int_A \omega_q$.

On the other hand, let G' be the Grassman manifold of $(n-q-1)$ -dimensional planes in P^{n-1} . Since S is an analytic set in P^{n-1} of dimension at most $q-1$, from Lemma A.1 of [2], there exists a thin analytic set T' in G' such that for all $M' \in G'-T'$, we have $M' \cap S = \emptyset$. Thus we may choose an $(n-q-1)$ -dimensional plane M' in P^{n-1} such that $M' \cap S = \emptyset$ and $\deg B = \#\{M' \cap B\}$. Let $M = \pi^{-1}(M') \cup \{a\}$. Then M is an $(n-q)$ -dimensional plane in P^n such that $\#\{A \cap M\} < \infty$, and each point of $M' \cap B$ has exactly λ distinct pre-images in $M \cap A$. Thus, again using the induction assumption, we have $c(q)\int_A \omega_q = \lambda c(q)\int_B \omega'_q = \lambda(\#\{M' \cap B\}) = \#\{M \cap A\} \leq \deg A$.

Hence $c(q)\int_A \omega_q = \deg A$.

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