ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO SOME nTH ORDER LINEAR DIFFERENTIAL EQUATIONS¹

I. NORMAN KATZ

Let $Lu \equiv u^{(n)} + \sum_{j=0}^{n-2} a_j(t)u^{(j)}$, where $a_j(t)$, $j=0, \dots, n-2$, are continuous on $[t_0, \infty)$, and let $z_j(t)$, $j=1, \dots, n$, be linearly independent solutions to

$$(1) \quad Lz = 0.$$

We study the behavior as $t \rightarrow \infty$ of solutions to

$$(2) \quad Lu = \sum_{j=0}^{n-2} g_j(t)u^{(j)}$$

where $g_j(t)$, $j=0, \dots, n-2$, are continuous on $[t_0, \infty)$, in terms of the known functions $z_1(t), \dots, z_n(t)$. Specifically, we seek conditions in order that for each solution $u(t)$ of (2), there exist constants A_j , $j=1, \dots, n$, such that

$$(3) \quad u^{(i)}(t) = A_1(t)z_1^{(i)}(t) + \dots + A_n(t)z_n^{(i)}(t), \quad i = 0, \dots, n-1,$$

and

$$(4) \quad \lim_{t \rightarrow \infty} A_j(t) = A_j, \quad \int_{t_0}^{\infty} |A'_j(t)| dt < \infty, \quad j = 1, \dots, n,$$

hold; and that for each choice of A_j there exists a solution $u(t)$ of (2) satisfying (3) and (4). For bounded functions $z_j^{(k-1)}(t)$, $j=1, \dots, n$, $k=1, \dots, n$, this is stronger than the case where (1) and (2) are *asymptotically equivalent*, i.e., for each solution $u(t)$ of (2) there is a solution $z(t)$ of (1) such that $|u^{(i)}(t) - z^{(i)}(t)| \rightarrow 0$ as $t \rightarrow \infty$, $i=0, \dots, n-1$, and conversely (see [3]).

Denote by $W(t) = W(z_1(t), \dots, z_n(t))$ the Wronskian of $z_j(t)$, $j=1, \dots, n$, and let $W_j(t)$, $j=1, \dots, n$, be the determinant obtained by replacing all elements in the j th column of $W(t)$ by zero except for the element in the n th row which is replaced by 1. Let

$$y(t) = \max_{k=1, \dots, n} \left| \sum_{i=0}^{n-2} g_i(t)z_k^{(i)}(t) \right|.$$

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We prove the following theorem.

THEOREM. *Given arbitrary constants $A_j, j=1, \dots, n$, there exists a solution to (2) which can be written in the form (3) with $A_j(t), j=1, \dots, n$, satisfying (4) if and only if*

$$(5) \quad \sum_{j=1}^n \int_{t_0}^{\infty} |W_j(t)| y(t) dt < \infty.$$

Also, every solution to (2) can be written in the form (3) with $A_j(t)$ satisfying (4) (for some finite $A_j, j=1, \dots, n$) if and only if (5) is satisfied.

REMARK 1. Our theorem is essentially a characterization of the method of variation-of-constants [1, p. 87], applied to (2) when the general solution to (1) is known. The absence of the $z^{(n-1)}$ term in (1) and the $u^{(n-1)}$ term in (2) implies that $W(z_1(t), \dots, z_n(t)) = W(z_1(t_0), \dots, z_n(t_0)) = W(t_0) \neq 0$ and $W(u_1(t), \dots, u_n(t)) = W(u_1(t_0), \dots, u_n(t_0)) \neq 0$ when $u_1(t), \dots, u_n(t)$ are linearly independent solutions to (2), and this is essential to our proof of the theorem.

When $n=2$, $|W_1(t)| = |z_2(t)|$, $|W_2(t)| = |z_1(t)|$, and it is easily verified that (5) is equivalent to

$$(6) \quad \int_{t_0}^{\infty} |g_0(t)| \max[|z_1(t)|^2, |z_2(t)|^2] dt < \infty.$$

This condition was shown to be sufficient by Trench [2]. (Trench [2] does not state that $\int_{t_0}^{\infty} |A'_j(t)| dt < \infty, j=1, 2$, when (6) is satisfied but this follows easily from his proof.) It follows from our theorem that (6) is also necessary if all solutions to (2) are required to satisfy $\int_{t_0}^{\infty} |A'_j(t)| dt < \infty, j=1, 2$. Other related results for arbitrary n are obtained in [3], [4], [5], and [6].

REMARK 2. *In the special case when $a_j(t) \equiv 0, j=0, \dots, n-2$, (2) becomes*

$$(7) \quad u^{(n)} = \sum_{i=0}^{n-2} g_i(t) u^{(i)}.$$

Now we have $z_k(t) = t^{k-1}, k=1, \dots, n$, and it is easily shown that $W_j(t) = O(t^{n-j})$ as $t \rightarrow \infty, j=1, \dots, n$. From our theorem it follows that if $\int_{t_0}^{\infty} |g_i(t)| t^{2n-2-i} dt < \infty, i=0, \dots, n-2$, then every solution to (7) can be written in the form $u(t) = \sum_{k=1}^n A_k(t) t^{k-1}, u^{(i)}(t) = \sum_{k=i+1}^n (k-1) \dots (k-i) A_k(t) t^{k-1-i}, i=1, \dots, n-1$ where $\lim_{t \rightarrow \infty} A_k(t), k=1, \dots, n$, may be prescribed.

PROOF. Suppose $u_k(t)$, $k=1, \dots, n$, are linearly independent solutions to (2) which can be written in the form

$$u_k^{(i)}(t) = \sum_{j=1}^n A_{jk}(t) z_j^{(i)}(t), \quad i = 0, \dots, n-1,$$

with $\int_{t_0}^{\infty} |A'_{jk}(t)| dt < \infty$, $j=1, \dots, n$, $k=1, \dots, n$, (which implies that $\lim_{t \rightarrow \infty} A_{jk}(t) = A_{jk}$ exists and is finite). Let the matrix $A(t) \equiv [A_{jk}(t)]$ and let the matrix $A \equiv [A_{jk}]$. Then

$$(8) \quad W(u_1(t), \dots, u_n(t)) = W(z_1(t), \dots, z_n(t)) \det A(t).$$

Since $W(u_1(t), \dots, u_n(t)) = W(u_1(t_0), \dots, u_n(t_0)) \neq 0$ and $W(z_1(t), \dots, z_n(t)) = W(z_1(t_0), \dots, z_n(t_0)) \neq 0$ it follows that $\det A(t) = \det A \neq 0$. Now given arbitrary constants A_j , $j=1, \dots, n$, let B_k , $k=1, \dots, n$, be determined from $\sum_{k=1}^n A_{jk} B_k = A_j$, $j=1, \dots, n$, and let $A_j(t) = \sum_{k=1}^n A_{jk}(t) B_k$. Then $\lim_{t \rightarrow \infty} A_j(t) = A_j$ and $\int_{t_0}^{\infty} |A'_j(t)| dt < \infty$, $j=1, \dots, n$. Also

$$\begin{aligned} u^{(i)}(t) &= \sum_{k=1}^n B_k u_k^{(i)}(t) = \sum_{j=1}^n \left(\sum_{k=1}^n A_{jk}(t) B_k \right) z_j^{(i)}(t) \\ &= \sum_{j=1}^n A_j(t) z_j^{(i)}(t), \quad i = 0, \dots, n-1, \end{aligned}$$

is in the form (3) and satisfies (2). Thus, if we show that (5) is a necessary and sufficient condition for the existence of n linearly independent solutions to (2) which can be written in the form (3) satisfying (4), this will show that a solution $u(t)$ to (2) exists which can be written in the form (3) when A_j , $j=1, \dots, n$, are arbitrary constants. Also, any solution to (2) can be written as a linear combination of n linearly independent solutions. Therefore, we need only to prove the theorem for linearly independent solutions $u_k(t)$, $k=1, \dots, n$.

Let $u(t) = \sum_{j=1}^n A_j(t) z_j(t)$ be a solution to (2) satisfying (3). Then it follows that

$$\begin{aligned} \sum_{j=1}^n A'_j(t) z_j^{(i)}(t) &= 0, \quad i = 0, \dots, n-2, \\ (9) \quad \sum_{j=1}^n A'_j(t) z_j^{(n-1)}(t) &= \sum_{i=0}^{n-2} g_i(t) u^{(i)}(t) \\ &= \sum_{j=1}^n \left(\sum_{i=0}^{n-2} g_i(t) z_j^{(i)}(t) \right) A_j(t). \end{aligned}$$

Conversely, if $A_j(t)$, $j=1, \dots, n$, satisfy (9) then $u(t)$ is a solution to (2).

We now show that if (5) is satisfied, and $A_j(t)$, $j=1, \dots, n$, satisfy (9), then

$$\int_{t_0}^{\infty} |A_j'(t)| dt < \infty$$

(which implies also that $\lim_{t \rightarrow \infty} A_j(t)$ exists and is finite). From (9) we have

$$(10) \quad A_j'(t) = (W_j(t)/W(t_0)) \sum_{k=1}^n A_k(t) \sum_{i=0}^{n-2} g_i(t) z_k^{(i)}(t), \quad j = 1, \dots, n,$$

$$A_j(t) = A_j(t_0) + (1/W(t_0)) \int_{t_0}^t \sum_{k=1}^n A_k(s) W_j(s) \sum_{i=0}^{n-2} g_i(s) z_k^{(i)}(s) ds,$$

$$j = 1, \dots, n,$$

and

$$|A_j(t)| \leq |A_j(t_0)| + (1/|W(t_0)|) \cdot \int_{t_0}^t \sum_{k=1}^n |A_k(s)| |W_j(s)| \left| \sum_{i=0}^{n-2} g_i(s) z_k^{(i)}(s) \right| ds$$

$$\leq |A_j(t_0)| + (1/|W(t_0)|) \int_{t_0}^t \sum_{k=1}^n |A_k(s)| |W_j(s)| y(s) ds,$$

$$j = 1, \dots, n.$$

Summing over j we obtain

$$\sum_{j=1}^n |A_j(t)| \leq \sum_{j=1}^n |A_j(t_0)| + (1/|W(t_0)|) \cdot \int_{t_0}^t \left(\sum_{k=1}^n |A_k(s)| \right) \left(\sum_{j=1}^n |W_j(s)| y(s) \right) ds.$$

Using Bellman's lemma [7, p. 35], we have

$$(11) \quad \sum_{j=1}^n |A_j(t)| \leq \left(\sum_{j=1}^n |A_j(t_0)| \right) \exp \left((1/|W(t_0)|) \int_{t_0}^t \sum_{j=1}^n |W_j(s)| y(s) ds \right).$$

Since (5) is satisfied, the integral in (11) is convergent as $t \rightarrow \infty$ which

implies that $A_j(t)$, $j=1, \dots, n$ are bounded. This used in (10) gives

$$\begin{aligned} \int_{t_0}^t |A'_j(s)| ds &\leq (1/|W(t_0)|) \int_{t_0}^t \sum_{k=1}^n |A_k(s)| |W_j(s)| \\ &\quad \cdot \left| \sum_{i=0}^{n-2} g_i(s) z_k^{(i)}(s) \right| ds \\ &\leq (1/|W(t_0)|) \int_{t_0}^t \sum_{k=1}^n (|A_k(s)| |W_j(s)| y(s) ds, \\ &\quad j = 1, \dots, n, \end{aligned}$$

which shows that $\int_{t_0}^\infty |A'_j(t)| dt < \infty$, $j=1, \dots, n$. Now let $A_{jk}(t)$ be the solution of (9) with $A_{jk}(t_0) = \delta_{jk}$, $j=1, \dots, n$, $k=1, \dots, n$, and let

$$u_k(t) = \sum_{j=1}^n A_{jk}(t) z_j(t), \quad k = 1, \dots, n.$$

Then $u_k(t)$, $k=1, \dots, n$, satisfies (2),

$$u_k^{(i)}(t) = \sum_{j=1}^n A_{jk}(t) z_k^{(i)}(t), \quad i = 0, \dots, n-1,$$

is in the form (3) with $A_{jk}(t)$ satisfying (4), and from (8) it follows that $u_k(t)$, $k=1, \dots, n$, are linearly independent. This completes the proof of the sufficiency of (5).

To prove the necessity of (5), assume that

$$u_k(t) = \sum_{j=1}^n A_{jk}(t) z_j(t), \quad k = 1, \dots, n,$$

are linearly independent solutions to (2) which satisfy (3) and (4). Then it follows from (8) that $\det A(t) = \det A(t_0) \neq 0$. Therefore, from (10), $A(t)$ is a fundamental matrix for the matrix equation

$$A' = Z(t)A$$

where the matrix

$$\begin{aligned} Z(t) &\equiv (1/(W(t_0))) \left[W_j(t) \sum_{i=0}^{n-2} g_i(t) z_k^{(i)}(t) \right], \\ &\quad j = 1, \dots, n, \quad k = 1, \dots, n. \end{aligned}$$

Thus

$$Z(t) = A'(t)A^{-1}(t) = (1/\det A(t))A'(t)B(t) = (1/\det A(t_0))A'(t)B(t),$$

where the elements of $B(t)$ are sums of products of the $A_{jk}(t)$. Since by assumption the $A_{jk}(t)$ approach finite limits as $t \rightarrow \infty$ the same is true for the elements of $B(t)$. Since by assumption

$$\int_{t_0}^{\infty} |A'_{jk}(t)| dt < \infty, \quad j = 1, \dots, n, \quad k = 1, \dots, n,$$

it now follows that the elements of $Z(t)$ are absolutely integrable over $[t_0, \infty)$, that is

$$\int_{t_0}^{\infty} |W_j(t)| \left| \sum_{i=0}^{n-2} g_i(t) z_k^{(i)}(t) \right| dt < \infty, \quad j = 1, \dots, n, \quad k = 1, \dots, n.$$

It is easily verified that this implies (5).

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WASHINGTON UNIVERSITY