A FIXED POINT THEOREM FOR OPEN Q -ACYCLIC n -MANIFOLDS¹

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Henderson and Livesay [4] proved a generalization of the Brouwer fixed point theorem by making use of a retraction theorem of Bing [1, Theorem 3]. Bing has extended his retraction theorem in [2]. It is the purpose of this note to demonstrate that Bing's extended result leads to a considerably more general fixed point theorem.

A noncompact manifold without boundary is called *open*. By ANR we mean an absolute neighborhood retract for the category of separable metric spaces. A space Y is *Q -acyclic* if $\tilde{H}_*(Y; Q) = 0$ (reduced singular homology with rational coefficients). A map $f: X \rightarrow Y$ is *compact* if the closure of $f(X)$ is compact.

THEOREM. *If M is an open Q -acyclic n -manifold, X is an $(n-2)$ -connected ANR imbedded as a closed subset of M , and U is a component of $M-X$ whose closure is not compact, then every compact map $f: (M-U, X) \rightarrow (M, M-U)$ has a fixed point.*

PROOF. By Bing's Retraction Theorem [2], there is a retraction r of $U \cup X$ onto X which, obviously, extends to a retraction $r: M \rightarrow M-U$. Since a retract of an ANR is an ANR and a retract of a Q -acyclic space is Q -acyclic, $M-U$ is a Q -acyclic ANR. Therefore, if $f: (M-U, X) \rightarrow (M, M-U)$ is a compact map, then, by the generalization of the Lefschetz fixed point theorem in [3], the compact map $rf: M-U \rightarrow M-U$ has a fixed point x_0 . If $f(x_0) \in U$, then $x_0 = rf(x_0) \in X$ which contradicts the hypothesis $f(X) \subseteq M-U$, so $f(x_0) \in M-U$ and therefore $x_0 = rf(x_0) = f(x_0)$.

COROLLARY (HENDERSON AND LIVESAY [4]). *Let X be an $(n-2)$ -connected compact ANR contained in a compact n -manifold, M , which is a rational homology n -sphere. If E is a component of $M-X$ and $p \in E$, then each map $f: (M-E, X) \rightarrow (M-p, M-E)$ has a fixed point.*

PROOF. Note that X is closed in the open n -manifold $M-p$ because X is compact and that f is a compact map because $M-E$ is compact. Thus the corollary is immediate once we establish that $M-p$ is

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Q -acyclic. By excision and hypothesis, $\tilde{H}_i(M, M-p; Q) = \tilde{H}_i(M; Q) = 0$ for $i \neq n$ so the exact sequence of the pair $(M, M-p)$ implies that $\tilde{H}_i(M-p; Q) = 0$ for $i \neq n-1, n$. Since $M-p$ is an open n -manifold, $\tilde{H}_n(M-p; Q) = 0$ by Lemma D of [5]. Therefore, the inclusion induced homomorphism $\tilde{H}_n(M; Q) \rightarrow \tilde{H}_n(M, M-p; Q)$ is a monomorphism from one one-dimensional vector space over Q to another and hence an isomorphism. The exact sequence now tells us that $\tilde{H}_{n-1}(M-p; Q) \cong \tilde{H}_{n-1}(M; Q) = 0$.

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