PURE STATES AND APPROXIMATE IDENTITIES¹

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1. Introduction. In this note we show that each norm separable C^* -algebra has an increasing abelian approximate identity and that, if the algebra has an identity, each pure state is multiplicative on some maximal abelian subalgebra.

Let A be a C^* -algebra without an identity element. A net $\{u_i\}_{i \in I} \subseteq A$, where I is a directed index set, is called an *approximate identity* for A if $\|u_i\| \leq 1$ for all $i \in I$, and $\|u_i x - x\| \rightarrow 0$; $\|x u_i - x\| \rightarrow 0$ for all $x \in A$. We say that $\{u_i\}_{i \in I}$ is *increasing* if $u_i \geq 0$ and $i \leq j \Rightarrow u_i \leq u_j$ for all $i, j \in I$. With u_i selfadjoint, if one of the limits exists, so does the other. Each C^* -algebra has an increasing approximate identity [2, 1.7.2]. An approximate identity $\{u_i\}_{i \in I}$ is *countable* if I is countable. It is *abelian* if u_i and u_j commute for all $i, j \in I$.

An element $x \in A$ is said to be *strictly positive* if $\rho(x) > 0$ for each nonzero positive linear functional ρ on A . A strictly positive element is positive [2, 2.6.2].

We use the following notation: If M is a collection of vectors in a Hilbert space H , and $\mathfrak{F}$ is a family of bounded linear operators on H , then $[\mathfrak{F}M]$ is the closed linear span of the set $\{F\xi: F \in \mathfrak{F}, \xi \in M\}$.

2. Results.

LEMMA 1. *If $x \in A$ is strictly positive, and π is a nondegenerate representation of A on a Hilbert space H , then $[\pi(x)H] = H$.*

PROOF. Suppose $0 \neq \xi \in [\pi(x)H]^\perp$. Since π is nondegenerate there is an element $a \in A$ such that $\pi(a)\xi \neq 0$. Let $\rho = \omega_\xi \circ \pi$, where ω_ξ is the positive linear functional $y \rightarrow (y\xi, \xi)$. Then $\rho(a*a) = (\pi(a*a)\xi, \xi) = \|\pi(a)\xi\|^2 \neq 0$, so ρ is a positive, nonzero linear functional on A . But $\rho(x) = (\pi(x)\xi, \xi) = 0$ which contradicts the assumption that x is strictly positive. Hence $[\pi(x)H]^\perp = (0)$ and the lemma follows.

THEOREM 1. *A C^* -algebra A has a countable increasing abelian approximate identity if and only if A contains a strictly positive element.*

PROOF. If $x_0 \in A$ is strictly positive, we may take x_0 with norm equal to 1. Let $u_i = x_0^{1/i}$, and observe that $u_i \geq 0$, $\|u_i\| = 1$; $i \geq j \Rightarrow u_i \geq u_j$ and u_i and u_j commute for all $i, j \in I$. We want to show that for any

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$x \in A$, $\|xu_i - x\| \rightarrow 0$. It is sufficient to do this for $x \geq 0$. (Using Theorem 5.1 of Akemann [1], the present proof could be somewhat shortened, but we prefer to give a complete argument.)

Let $x \geq 0$, and let y be the unique positive square root of x . Let $\tilde{A}$ be the C^* -algebra obtained by adjoining an identity e to A . Then $u_i \leq e$ for all i , so that $0 \leq yu_i y \leq yey = x$, and $yu_i y \leq yu_j y$ if $i \leq j$. Hence, if $z_i = x - yu_i y$, $\{z_i\}$ becomes a monotone decreasing sequence of positive elements in A . We claim that $\|z_i\| \rightarrow 0$. Let

$$S = \{\rho \in A^*: \rho \geq 0; \|\rho\| \leq 1\}.$$

S is compact in the w^* -topology [2, 2.5.5]. We may regard each z_i as a continuous function on S by the evaluation map. Since $z_i \geq 0$, $\|z_i\| = \sup\{\rho(z_i): \rho \in S\}$ [2, 2.7.3]; so that it suffices to show that z_i converges uniformly to 0 on S . As the sequence z_i is monotone, this will follow from Dini's theorem once we know that $\rho(z_i) \rightarrow 0$ for each $\rho \in S$.

Let π be a nondegenerate representation of A on a Hilbert space H . $\pi(u_i) = \pi(x_0^{1/i}) = \pi(x_0)^{1/i}$, which by spectral theory converges strongly to the range projection of $\pi(x_0)$. Since x_0 is strictly positive it follows by Lemma 1 that $\pi(u_i) \rightarrow I$ strongly on H , where I is the identity operator on H .

Let $\rho \neq 0$ be an arbitrary element of S and π_ρ be the associated representation of A on the Hilbert space H_ρ . Then π_ρ is nondegenerate with a cyclic vector ξ_ρ and

$$\begin{aligned} \rho(z_i) &= (\pi_\rho(z_i)\xi_\rho, \xi_\rho) \\ &= ((\pi_\rho(x) - \pi_\rho(yu_i y))\xi_\rho, \xi_\rho) \\ &= (\pi_\rho(y)\xi_\rho - \pi_\rho(u_i)\pi_\rho(y)\xi_\rho, \pi_\rho(y)\xi_\rho) \end{aligned}$$

which converges to zero since $\pi_\rho(u_i) \rightarrow I$ strongly. Hence $\|z_i\| \rightarrow 0$.

Working in $\tilde{A}$ (as Akemann does in [1]), let v_i be the positive square root of $e - u_i$. Then $\|yv_i\|^2 = \|yv_i v_i y\| = \|y(e - u_i)y\| = \|x - yu_i y\| \rightarrow 0$, and hence $\|xu_i - x\| = \|y^2 v_i^2\| \leq \|y\| \cdot \|yv_i\| \cdot \|v_i\| \leq \|y\| \cdot \|yv_i\| \rightarrow 0$. Thus $\{u_i\}$ is an approximate identity.

Conversely, suppose $\{u_i\}$ is an increasing abelian approximate identity, and let $x = \sum_{n=1}^{\infty} 2^{-n} u_n$. If ρ is a nonzero positive linear functional on A , we know that $\rho(u_n) \rightarrow \|\rho\|$ [2, 2.1.5]. Hence $\rho(u_n) > 0$ for some n , so $\rho(x) = \sum_{n=1}^{\infty} 2^{-n} \rho(u_n) > 0$. This shows that x is strictly positive, and the proof is complete.

Observe that if A is separable, then A has a strictly positive element. Indeed, if $\{y_n\}$ is dense in A , then $\{x_n = y_n^* y_n\}$ is dense in $A^+ = \{x \in A: x \geq 0\}$. Clearly $x = \sum_{n=1}^{\infty} (2^n \|x_n\|)^{-1} x_n$ is strictly positive in A .

COROLLARY 1. *Any separable C^* -algebra has a countable increasing abelian approximate identity.*

REMARK. Let X be a locally compact Hausdorff space, $A = \mathcal{C}^0(X)$, the C^* -algebra of all continuous complex functions on X vanishing at infinity. It is easily verified that A contains a function f which is everywhere positive if and only if X is σ -compact. Since each state on A may be represented by a Borel measure on X , we see that such a function f is a strictly positive element of A . Evidently X may be σ -compact without having a countable base for its topology, so A may have a strictly positive element without being separable. Needless to say, A will not always have strictly positive elements. An example is $\mathcal{C}^0(\mathbf{R})$, when $\mathbf{R}$ is given the discrete topology.

A positive linear functional ρ on a C^* -algebra A is a *state* if $\|\rho\| = 1$. If A has an identity e , this is equivalent to the condition $\rho(e) = 1$. We say ρ is *pure* if $\rho \neq 0$ and each positive, linear functional γ on A such that $0 \leq \gamma \leq \rho$, is of the form $\gamma = a\rho$; $0 \leq a \leq 1$.

THEOREM 2. *Let A be a separable C^* -algebra with identity. If ρ is a pure state on A , then there is a maximal abelian C^* -subalgebra B of A such that $\rho|_B$ is multiplicative.*

PROOF. Let N_ρ be $\{x \in A : \rho(x^*x) = 0\}$ and A_0 be $N_\rho \cap N_\rho^*$. A_0 is a C^* -subalgebra of A and is therefore separable. Hence A_0 contains a strictly positive element x_0 . Let B_0 be a maximal abelian C^* -subalgebra of A_0 containing x_0 and B be $B_0 + \mathbf{C} \cdot e$. Then B is an abelian C^* -subalgebra of A . Since ρ vanishes on B_0 , $\rho|_B$ is multiplicative of norm 1. To show that B is a maximal abelian C^* -subalgebra of A , it suffices to show that if a selfadjoint x in A commutes with B , then $x \in B$. Now, $x \in B$ if and only if $x - \rho(x)e \in B$; so we may assume that $\rho(x) = 0$. Let π_ρ be the irreducible representation of A associated with ρ on the Hilbert space H_ρ , with cyclic vector ξ_ρ [2, 2.5.4]. Let $H_0 = [\xi_\rho]^\perp$. We claim that $[\pi_\rho(A_0)H_\rho] = H_0$. Indeed, if $y \in A_0$ then $\|\pi_\rho(y)\xi_\rho\|^2 = (\pi_\rho(y^*y)\xi_\rho, \xi_\rho) = \rho(y^*y) = 0$; so that $(\pi_\rho(y)\pi_\rho(x)\xi_\rho, \xi_\rho) = 0$ for all x in A . On the other hand, let ξ in H_0 be arbitrary. By the transitivity theorem [2, 2.8.3] there is a selfadjoint element $y \in A$ such that $\pi_\rho(y)\xi_\rho = 0$ and $\pi_\rho(y)\xi = \xi$. But then $y \in A_0$ and the claim follows. Hence $\pi_\rho|_{A_0}$ is a nondegenerate representation on H_0 . Let E be the orthogonal projection of H_ρ onto H_0 . By Lemma 1, $[\pi_\rho(x_0)H_0] = H_0$. Now x_0 and x commute; so that E and $\pi_\rho(x)$ commute. Hence H_0 and $[\xi_\rho]$ are invariant under $\pi_\rho(x)$. This means that ξ_ρ is an eigenvector for $\pi_\rho(x)$; so $\pi_\rho(x)\xi_\rho = a\xi_\rho$ for some real a . Now $0 = \rho(x) = (\pi_\rho(x)\xi_\rho, \xi_\rho) = a(\xi_\rho, \xi_\rho) = a$. Hence $\pi_\rho(x)\xi_\rho = 0$ so $x \in A_0$. Since B_0 is

maximal abelian in A_0 it follows that $x \in B_0 \subseteq B$. The proof is complete.

REFERENCES

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