HOLOMORPHIC FUNCTIONS WITH DENSE SETS OF PLESSNER POINTS

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Let the function f be defined in the unit disk $D = \{z: |z| < 1\}$. A point $e^{i\theta}$ on the unit circle C is called a *Plessner point* of f provided each angular cluster set of f at $e^{i\theta}$ coincides with the extended plane. In this note we answer the following question: if the set of Plessner points of a holomorphic function is dense on C , must it automatically satisfy some measure-theoretic condition?

THEOREM. *For each number α ($0 \leq \alpha \leq 2\pi$) there exists a holomorphic function in D whose set of Plessner points is dense on C and has measure α .*

PROOF. First we deal with the case where $\alpha = 0$, and we begin with the construction of a locally univalent function having exactly one Plessner point.

Let G be a simply connected Riemann surface, of finite area, and consisting of the unit disk together with a smooth vermiform appendage that winds its way over the plane in such a way that each point of the plane is a boundary point of G . Let the function f map the disk D conformally onto G , and let it map the origin onto the point $w = 0$ in the portion of G that covers the unit disk in the w -plane. Then there exists exactly one point $e^{i\theta}$ such that the image of each curve in D from 0 to $e^{i\theta}$ passes through the full length of the appendage. The cluster set of f relative to each curve in D that terminates at $e^{i\theta}$ is the extended plane; therefore $e^{i\theta}$ is a Plessner point of f .

By adding further appendages to G , each appropriately branched, we obtain a simply connected Riemann surface G^* such that every continuous one-to-one mapping from D to G^* has a dense set of Plessner points on C . If we observe the precaution of keeping the area of G^* finite, then each conformal mapping f^* from D to G^* has a set of Fatou points of measure 2π , hence a set of Plessner points of measure 0.

The case where $\alpha = 2\pi$ is trivial, as we see by examining the function

$$g(z) = \sum k_n z^{k_n},$$

where $k_{n+1}/k_n \rightarrow \infty$. Since $|g(z)| > k_n/4$ at all points on the circle

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$|z| = 1 - 1/k_n$ ($n > n_0$), the set of Fatou points of g has measure 0 [2, p. 212], and therefore its set of Plessner points has measure 2π [1, p. 220].

To complete the proof, suppose that $0 < \alpha < 2\pi$. Let h map D conformally onto the right half of the unit disk in the w -plane in such a way that the pre-image of the circular part of the boundary of the half-disk has length α . Clearly, the set of Plessner points of the composite function $g^* = gh$ (where g denotes the lacunary function in the preceding paragraph) has measure α . Finally, let f^* be a conformal mapping of D onto G^* , and let $F = f^* + g^*$. If $e^{i\theta}$ is a Plessner point of f^* and a Fatou point of g^* (or vice versa), then it is a Plessner point of F . Therefore F has the properties described in the theorem.

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