A NOTE ON SEIFERT CIRCLES

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Introduction. It is known that an oriented polygonal knot spans an orientable surface in E^3 [3]. To construct such a surface, we first move the knot into a regular position; we define a collection of unknotted simple closed curves called Seifert circles; finally we attach a disjoint collection of disks to the Seifert circles. There is nothing in this construction that allows one to claim that the Seifert circles do not “nest.” If the Seifert circles are nonnested, then there is a projection of the surface into a plane which has singularities only at a finite number of points, i.e. below the crossing points of the knot. R. H. Fox [1] noted that by changing the knot diagram in particular cases he could have the Seifert circles nonnested. The following theorem states that this can always be done.

THEOREM 1. *If K is an oriented polygonal knot in E^3 , then there is a homeomorphism of E^3 onto itself taking K to a position which has non-nested Seifert circles.*

Before proceeding with the proof we give the following definitions. We will always assume that the knot is in regular position with respect to the xy plane in E^3 [2]. At each crossing we will span a “twisted” rectangle with an oriented boundary so that a pair of opposite sides of the rectangle will be attached with opposite orientation to the undercrossing and overcrossing segments (Figure 1).

FIGURE 1

Let K_* be the union of K and the “twisted” rectangles. Hence $K_* - [(interior\ of\ the\ attached\ rectangles) \cup (interior\ of\ the\ attached\ segments)]$ is a collection of simple closed curves which we will call *Seifert circles*. The Seifert circles obtained from a representation of K will be called *nonnested* if when they are projected into the xy plane

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the bounded complementary domain of each projected Seifert circle contains no other projected Seifert circle.

An orientable surface which has a simple closed curve as its boundary may be thought of as a disk with $2n$ attached bands so that the unattached sides of each band have opposite orientation. Hence we may assume that K is the boundary of the unit disk in the xy plane with $2n$ attached bands, $B_1, B_2, \dots, B_{2n}$, satisfying the following properties:

1. No part of B_i lies above or below the unit disk.
2. The width of each B_i is less than $\epsilon/4$.
3. If c_i is the centerline of B_i , then $\cup c_i$ is in regular position with respect to the xy plane.
4. If a and b are crossing points in the projection of $\cup c_i$ into the xy plane, then the distance from a to b is greater than ϵ .

PROOF OF THEOREM 1. Let $abcd$ be a segment of some B_i such that ab and cd are segments of K and the projection of $abcd$ into the xy plane is a rectangle. Define h as a homeomorphism of E^3 onto itself which moves ab to cd and cd to ab by twisting B_i at ad and bc and is the identity outside a small neighborhood of $abcd$ (Figure 2).

FIGURE 2

Step 1. Apply h to a small segment at each end of B_i and move all "twists" of B_i to a small segment at one end of B_i . Thus when K is projected, there will be a projected Seifert circle which contains the unit disk with no other projected Seifert circles in its bounded domain.

Step 2. We note that there are two types of band crossings, i.e. Figure 3 or Figure 4. If a band crossing occurs as in Figure 3, leave it alone. If a band crossing occurs as in Figure 4, then apply h to one of the bands.

FIGURE 3

FIGURE 4

Apply Step 2 at each band crossing to obtain the desired representation of K . Q.E.D.

REFERENCES

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