A RECURSION FORMULA FOR FINITE PARTITION LATTICES

TERRENCE J. BROWN

A *partition* on a nonempty set S is a collection of disjoint nonempty subsets, called blocks, whose union is S. The recursion formula

$$B_n = \sum_{j=0}^{n-1} \binom{n-1}{j} B_j \qquad (n \geqq 1,\ B_0 = 1) \tag{1}$$

for the number of partitions on a finite set with n elements is well known (cf. Rota [4]). We obtain equation (2) which generalizes (1) and which yields a derivation of the Möbius function for partition lattices different from those in [2], [3] and [5].

Let S be a finite nonempty set. Two partitions σ and π of S satisfy $\sigma \leqq \pi$ if every block of σ is contained in a block of π. $\leqq$ is a partial ordering of the collection $L(S)$ of partitions of S and $(L(S), \leqq)$ is a lattice (cf. Birkhoff [1]). If h is a function from the natural numbers into the integers Z, it is possible to define a function $k: L(S) \to Z$ by setting $k(\sigma) = \prod_{B \in \sigma} h(|B|)$ for $\sigma \in L(S)$ ($|B|$ = the cardinality of the block B). It is then possible to define another function H from the nonnegative integers into Z by setting $H(0) = 1$ and $H(n) = \sum_{\sigma \in L_n} k(\sigma)$ for $n \geqq 1$, where L_n is the lattice of partitions on a set with n elements.

THEOREM. *If h, k and H are as above then*

$$H(n) = \sum_{j=0}^{n-1} \binom{n-1}{j} h(j+1) H(n-1-j). \tag{2}$$

PROOF. Assume now that finite nonempty S has n elements. First we show that if C is a nonempty subset of S, then $\sum_{C \in \sigma \in L(S)} k(\sigma) = h(|C|)H(|S-C|)$. $C = S$ implies $\sum_{C \in \sigma \in L(S)} k(\sigma) = k\{S\} = h(|S|)H(0)$. If $C \subsetneqq S$ then

$$\begin{aligned} \sum_{C \in \sigma \in L(S)} k(\sigma) &= \sum_{C \in \sigma \in L(S)} h(|C|) k(\sigma - \{C\}) \\ &= h(|C|) \sum_{\pi \in L(S-C)} k(\pi) = h(|C|) H(|S-C|). \end{aligned}$$

Received by the editors September 25, 1968.

Because S is nonempty we may pick $a \in S$. Define f: $L(S) \to P(S - \{a\})$ (the power set of $S - \{a\}$) by $f(\sigma) = \sigma_a - \{a\}$, $\sigma \in L(S)$, where σ_a is the block of σ containing a. For

$$D \in P(S - \{a\}) f^{-1}(D) = \{\sigma \in L(S) \mid D \cup \{a\} \in \sigma\}.$$

As f partitions $L(S)$,

$$\begin{aligned} H(n) &= \sum_{\sigma \in L(S)} k(\sigma) = \sum_{D \in P(S - \{a\})} \sum_{\sigma \in f^{-1}(D)} k(\sigma) \\ &= \sum_{j=0}^{n-1} \sum_{D \in P(S-\{a\}); |D| = j} \sum_{D \cup \{a\} \in \sigma \in L(S)} k(\sigma) \\ &= \sum_{j=0}^{n-1} \sum_{D \in P(S-\{a\}); |D| = j} h(|D \cup \{a\}|). \end{aligned}$$

$$H(|S - D - \{a\}|) = \sum_{j=0}^{n-1} \binom{n-1}{j} h(j+1) H(n-1-j).$$

Equation (2) now follows.

Putting $h(n) = 1$, $n \geqq 1$, k maps each partition σ onto 1 and $H(n)$ becomes B_n while equation (2) reduces to (1).

The 0 of L_n is the partition having n blocks while the 1 of L_n is the partition having one block. See Rota [**3**] for the definition of the Möbius function μ.

COROLLARY. *If $\mu_n = \mu(0, 1)$ for the lattice L_n of partitions on a set with n elements, then* $\mu_n = (-1)^{n-1}(n-1)!$

PROOF. Set $h(n) = \mu_n$ for $n \geqq 1$. $\sigma \in L_n$ implies that

$$k(\sigma) = \prod_{B \in \sigma} h(|B|) = \prod_{B \in \sigma} \mu_{|B|} = \mu(0, \sigma)$$

(cf. Rota [**3**], especially Proposition 5, p. 345, and the lemma on p. 359). Since $H(m) = \sum_{\sigma \in Lm} k(\sigma) = \sum_{\sigma \in Lm} \mu(0, \sigma) = 0$ for $m \geqq 2$, $H(0) = 1$ and $H(1) = \mu_1 = 1$, we see from (2) that

$$0 = H(n) = \binom{n-1}{n-2} h(n-1) H(1) + \binom{n-1}{n-1} h(n) H(0),$$

i.e., that $\mu_n = -(n-1)\mu_{n-1}$ for $n \geqq 2$. From $\mu_1 = 1$ the corollary follows.

References

1. Garrett Birkhoff, *Lattice theory*, rev. ed., Amer. Math. Soc., Providence, R. I., 1966.

2. Roberto Frucht and Gian-Carlo Rota, *La función de Möbius para el retículo di particiones de un conjuncto finito*, Scientia; revista cientifica y técnica (Chile) N. **122** (1963), 111–115.

3. Gian-Carlo Rota, *On the foundations of combinatorial theory.* I, *Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie **2** (1964), 340–368.

4. ———, *The number of partitions of a set*, Amer. Math. Monthly **71** (1964), 498–504.

5. M. P. Schutzenberger, *Contribution aux applications statistiques de la théorie de l'information*, Publ. Inst. Statist. Univ. Paris **3** (1954), 5–117.

University of Missouri at Kansas City