IDEMPOTENT NOETHER LATTICES

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In his paper, *Abstract commutative ideal theory* [2], Dilworth proved that a Noether lattice on which the multiplication is the meet operation is a finite Boolean algebra. This note proves that if the multiplication in a Noether lattice is idempotent ($A^2=A$ for all A in the lattice), then the lattice is a finite Boolean algebra. In a Noether lattice the term maximal element refers to a maximal nonidentity element (i.e. a coatom).

THEOREM. *Let L be a Noether lattice in which the maximal elements are idempotent. Then L is a finite Boolean algebra.*

PROOF. Let M be a maximal element of L. Then $M=M^i$ for all i. This gives

$$\bigwedge_i (A \vee M^i) = A \vee M \quad \text{for all } A \in L.$$

Thus by Theorem 3.2 of [2], $A\vee M$ is the meet of all primary components of A contained in M. We thus have that each element of L is a meet of finitely many maximal elements of L. Since L is modular and every element of L is a meet of coatoms, L is complemented. Then by Theorem 7.31 of [1], L is a Boolean algebra. (The term "Noether lattice" has a different meaning in [1] from that in [2]. We are using the term as defined in [2].) Since L has the ascending chain condition, L is finite, and the theorem is proved.

Note that meet and multiplication coincide when multiplication is idempotent in a Noether lattice. To see this, observe that if

$$\begin{aligned}&(M_1 \wedge M_2 \wedge \cdots \wedge M_j)(N_1 \wedge N_2 \wedge \cdots \wedge N_k)\\ &\quad < M_1 \wedge \cdots \wedge M_j \wedge N_1 \wedge \cdots \wedge N_k,\end{aligned}$$

with M_i and N_i all maximal, then because each element of L is a meet of maximal elements, there exists another maximal element M such that

$$\begin{aligned}&(M_1 \wedge M_2 \wedge \cdots \wedge M_j)(N_1 \wedge N_2 \wedge \cdots \wedge N_k)\\ &\quad \leqq M \wedge M_1 \wedge \cdots \wedge M_j \wedge N_1 \wedge \cdots \wedge N_k.\end{aligned}$$

Then

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$$M_1M_2 \cdots M_jN_1N_2 \cdots N_k \leqq M.$$

Thus since M is prime, M_i or $N_i < M$ for some i, and this is impossible.

References

1. R. P. Dilworth and M. Ward, *Residuated lattices*, Trans. Amer. Math. Soc. **45** (1939), 335–354.

2. R. P. Dilworth, *Abstract commutative ideal theory*, Pacific J. Math. **12** (1962), 481–498.

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