ON SUBGROUPS OF FINITE SOLVABLE GROUPS

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In this note, the word "group" means "a finite solvable group." Let G be a group, and D a system normalizer of G. In [5] we introduced the subgroup $Q(D)$, generated by all subgroups of G in which D is subnormal. In this note we use one of the alternative characterizations of $Q(D)$, as given in [5], to define an analogue, $Q(H)$, for arbitrary subgroups H of G. We derive a covering-avoidance characterization of $Q(H)$, and deduce that it is homomorphism invariant. These results, in turn, can be used to shorten many of the proofs in [5].

We first recall some definitions. A Sylow system $\mathfrak{S}$ of G is said to *reduce into* H, if $\mathfrak{S}\cap H$ (i.e. the set of intersections of members of $\mathfrak{S}$ with H) is a Sylow system of H. An *H-composition-series of G* is a series

$$\{1\} = G_n \triangle G_{n-1} \triangle \cdots \triangle G_1 \triangle G_0 = G \tag{1}$$

in which each G_i is a maximal H-invariant normal subgroup of G_{i-1}. The groups G_i/G_{i+1} are referred to as *H-composition-factors* of G. If H induces (by conjugation) only the trivial automorphism on G_i/G_{i+1}, then the latter is *H-central*, otherwise it is *H-eccentric*. The product of the indices $|G_i: G_{i+1}|$, for those factors in (1) which are H-central and are avoided by H, is denoted by $z_0(H)$. Here a subgroup K *covers* G_i/G_{i+1} if $G_i\subseteq G_{i+1}K$, K *avoids* G_i/G_{i+1} if $G_i\cap K \subseteq G_{i+1}$. $z_0(H)$ is an invariant of H (and G), i.e. it does not depend on the series (1) (see [2]).

Let $\mathfrak{M}$ be a set of Sylow systems of G. We refer to $\mathfrak{M}$ as a *block*, if $\mathfrak{M}$ is disjoint from all of its conjugates (so that if we consider G as a permutation group on its Sylow systems, the conjugates of $\mathfrak{M}$ form an imprimitivity system).

Now let H be any subgroup of G. We denote by $\mathfrak{M}_0$ the smallest block of G which contains all the Sylow systems reducing into H.

DEFINITION. The stabilizer of $\mathfrak{M}_0$ (i.e. the set of all $g\in G$ such that $\mathfrak{M}_0^g=\mathfrak{M}_0$) is denoted by $Q(H)$.

THEOREM 1. *$Q(H)$ covers all H-central H-composition-factors of G. Moreover, if $K\supseteq H$ and K covers all H-central H-composition-factors, then $K\supseteq Q(H)$.*

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PROOF. Let G_i/G_{i+1} be an H-central factor in (1), and let $\mathfrak{S}$ be a Sylow system of G reducing into H. Then $\mathfrak{S}$ reduces into G_iH [**3**, Lemma 2.7]. Let D be $N_{G_iH}(\mathfrak{S}\cap G_iH)$. Then D transforms S into systems reducing into H (because they all have the same intersection with G_iH), and thus D stabilizes $\mathfrak{M}_0$, and $D\subseteq Q(H)$. Since D covers the central factor G_i/G_{i+1} of G_iH, $Q(H)$ covers G_i/G_{i+1}.

Now let $K\supseteq H$, and assume that K covers all H-central factors. A K-central factor is certainly H-central, so K covers all of its central factors, and thus K is abnormal (see [**2**, §2]; an abnormal subgroup is one for which $g\in\langle K, K^g\rangle$ for all $g\in G$). The intersections of K with the terms of (1) form an H-composition series of K, and as K covers all H-central factors in (1), these give rise to H-central factors of K of the same order. Thus $z_0(H)$, computed in K, is the same as $z_0(H)$, computed in G.

Let D be a system normalizer of G, and D_1 one of K. By [**2**, p. 541] there are $|H|/|D|\cdot z_0(H)$ Sylow systems of G reducing into H, $|H|/|D_1|\cdot z_0(H)$ systems of K reducing into H, and each system of K is the intersection with K of $|D_1|/|D|$ systems of G. It follows that the number of systems of G reducing into both K and H is

$$\frac{|D_1|}{|D|}\cdot\frac{|H|}{|D_1|}\cdot z_0(H) = \frac{|H|}{|D|}\cdot z_0(H)$$

i.e. all systems of G reducing into H reduce also into K. Let $\mathfrak{M}$ be the set of all Sylow systems reducible into K. Then, K being abnormal, $\mathfrak{M}$ is a block with stabilizer K [**5**, Lemma 2]. Thus $\mathfrak{M}\supseteq\mathfrak{M}_0$, and the stabilizer of $\mathfrak{M}_0$ is contained in the stabilizer of $\mathfrak{M}$.

REMARK 1. It is seen from the proof that it is enough to assume that K covers the H-central factors in a given series (1).

REMARK 2. For each central factor G_i/G_{i+1} in (1), let D_i be a system normalizer of G_iH, as in the first paragraph of the proof. Then we have seen that $D_i\subseteq Q(H)$, and that D_i covers G_i/G_{i+1}. Thus Theorem 1 implies that $Q(H)=\langle H, D_i\rangle$ (i ranges over all indices such that G_i/G_{i+1} is H-central).

REMARK 3. Take $K=Q(H)$ in the above proof. Then $\mathfrak{M}\supseteq_0.\mathfrak{M}$ If $\mathfrak{S}\in\mathfrak{M}_0$ and $g\in Q(H)$, then $\mathfrak{S}^g\in\mathfrak{M}_0$. Take $\mathfrak{S}$ to reduce into H, then we have seen that $\mathfrak{S}$ reduces into $Q(H)$, and all systems reducing into $Q(H)$ are conjugate under $Q(H)$ by [**1**, Lemma, p. 360]; thus $\mathfrak{M}\subseteq\mathfrak{M}_0$ and $\mathfrak{M}_0$ is the set of all Sylow systems reducing into $Q(H)$.

THEOREM 2. *Let $G\to G^*$ be an epimorphism, and let stars denote epimorphic images. Then $Q(H^*)=Q(H)^*$.*

PROOF. Let N be the kernel of the epimorphism, and let $R/N=Q(H)^*$, $Q=Q(H)$. We may assume that N is one of the terms in

(1). Then Q^* covers all H^*-central factors in the H^*-composition-series $\{G_i^*\}$ of G^*. Thus $Q^* \supseteq R$. In turn, R covers all H-central factors in (1), so $R \supseteq Q$, and $R^* = Q^*$.

Suppose $H \triangle\triangle L$, and let $\mathfrak{N}$ be the set of Sylow systems reducible into L. Then all systems in $\mathfrak{N}$ reduce into H, so $\mathfrak{N} \subseteq \mathfrak{M}_0$. As L stabilizes $\mathfrak{N}$, $L \subseteq Q(H)$. In general, $Q(H)$ is not generated by all such L, as we can see by taking H to be any self-normalizing subgroup that is not abnormal.

Now take D to be any subgroup normalizing the Sylow system $\mathfrak{S}$ of G. In the notations of Remark 2, $D \subseteq D_i$ for each of the i's considered there. Thus $Q(D) = \langle D_i \rangle$, and $D \triangle\triangle D_i$, as each D_i is nilpotent. So $Q(D)$ is generated by all subgroups in which D is subnormal. If $D \subseteq E$ and E is nilpotent, then $D \triangle\triangle E$, hence $E \subseteq Q(D)$. On the other hand, the subgroups D_i are nilpotent. We thus see that $Q(D)$ is, indeed, the subgroup introduced in [5], and at the same time we have alternative proofs for the properties of $Q(D)$ discussed there (the present treatment is slightly more general, as we allow D to be an arbitrary subgroup of a system normalizer).

As a further application, consider the problem: when is $\mathfrak{M}_0$ the set of all systems reducing into H? Suppose this is the case. By Remark 3, all systems of $Q(H)$ reduce into H, so that $H \triangle\triangle Q(H)$ [2] or [4]. We then have in $Q(H)$, and therefore also in G, $z_0(H) = |Q(H): H|$. Thus $Q(H)$ is the strong subnormalizer of H, in the sense of [5]. Conversely, assume that $H \triangle\triangle L$ and that $|L: H| = z_0(H)$. L covers or avoids all factors in (1), and the ones that L covers but H avoids must be H-central (they are H-isomorphic to factors between L and H). By orders, L covers all H-central factors, so $L \supseteq Q(H)$, $L = Q(H)$, and L is necessarily the strong subnormalizer of H. Since $H \triangle\triangle L$, all systems of L reduce into H, so $\mathfrak{M}_0$ is indeed the desired set of Sylow systems. We have thus reproved Theorems 3 and 4 of [5], while Theorem 5 there follows from our present Theorem 2.

References

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