A THEOREM ON THE SEMIGROUP OF BINARY RELATIONS

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The purpose of this note is to generalize a theorem of Zaretskii [2]. Notations and definitions used here are based on those of [1] and [2].

THEOREM. *Let X be an arbitrary set. The necessary and sufficient condition that the binary relation ρ is a regular element of the semigroup S_X is that $L(\rho)$ is a completely distributive complete lattice.*

PROOF. Necessity. Let $\rho=\rho\delta\rho$, where $\delta\in S_X$. Let $\sigma=\delta\rho$, then $\sigma^2=\sigma$ and $\rho=\rho\sigma$. It is known in [1] that $L(\sigma)$ and $L(\rho)$ are complete lattices in which joins are unions and, moreover, $L(\sigma)$ is completely distributive.

If $A\subseteq X$, then it is easy to show that $\psi(A)=\psi(\phi(A))$ and $\phi(A)=\chi(\psi(A))$ where $\phi(A)\in L(\sigma)$, $\psi(A)\in L(\rho)$ and $\chi(A)\in L(\delta)$. Define the mapping θ of $L(\sigma)$ onto $L(\rho)$ as follows: if $\phi(A)\in L(\sigma)$, then $\theta(\phi(A))=\psi(A)$. Clearly, θ preserves set-inclusion order and is one-to-one. Hence, $L(\sigma)$ is completely isomorphic with $L(\rho)$. This proves that $L(\rho)$ is completely distributive.

Sufficiency. Let $L(\rho)$ be a completely distributive complete lattice. Define the binary relation δ as follows: $(x, y)\in\delta$, iff $\rho(x, y)\rho\subset\rho$. Obviously, $\rho\delta\rho\subset\rho$.

For each $z\in X$, define $K_z=\{\psi(\{v\}): z\in\psi(\{v\})\}$. For any $y\in X$, let $K=\{K_z: z\in\psi(\{y\})\}$ and $S(K)$ denote the set of mappings s of $\psi(\{y\})$ into $L(\rho)$ such that for every $z\in\psi(\{y\})$, $s(z)\in K_z$. Then $\vee\{\wedge K_z: z\in\psi(\{y\})\}=\wedge\{\vee s(\psi(\{y\})): s\in S(K)\}$. Since lattice joins are unions, we have $\vee s(\psi(\{y\}))\supseteq\psi(\{y\})$, for each $s\in S(K)$, and hence $\wedge\{\vee s(\psi(\{y\})): s\in S(K)\}\supseteq\psi(\{y\})$. Therefore,

$$\cup\{\wedge K_z: z\in\psi(\{y\})\}=\vee\{\wedge K_z: z\in\psi(\{y\})\}\supseteq\psi(\{y\}).$$

Let $(x, y)\in\rho$. Then $x\in\psi(\{y\})$ and so there exists a $z\in\psi(\{y\})$ such that $x\in\wedge K_z$. Therefore $x\in\psi(\{w\})$ for some w satisfying

$$\psi(\{w\})\subseteq\wedge K_z\subseteq\cap K_z=\cap\{\psi(\{v\}): z\in\psi(\{v\})\}.$$

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But $w\delta z$ holds iff

$$\psi(\{w\}) \subseteq \bigcap \{\psi(\{v\}) : z \in \psi(\{v\})\}.$$

Thus, we have $x\rho w$, $w\delta z$ and $z\rho y$; hence $(x, y) \in \rho\delta\rho$. It follows that $\rho \subset \rho\delta\rho$. Therefore $\rho = \rho\delta\rho$. This completes the proof.

References

1. G. N. Raney, *A subdirect union representation for completely distributive complete lattices*, Proc. Amer. Math. Soc. **4** (1953), 518–522.

2. K. A. Zaretskii, *Regular elements of the semigroup of binary relations*, Uspehi Mat. Nauk **17** (1962), 177–179. (Russian)

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