ON THE GLEASON AND HARNACK METRICS FOR UNIFORM ALGEBRAS

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Let $B(X, R)$ and $B(X, C)$ consist of the bounded real and complex valued functions on the set X with $\|\cdot\|$ the supnorm on X. For $1 \in A \subset B(X, C)$ a complex subalgebra which separates the points of X one defines

$$G: G(u,v) = \operatorname{Sup}\{\,|f(u) - f(v)| : f \in A \text{ with } \|f\| \leqq 1\} \qquad \text{for } u, v \in X,$$

$$\sigma: \sigma(u, v) = \operatorname{Sup}\{\,|f(v)| : f \in A \text{ with } \|f\| \leqq 1 \text{ and } f(u) = 0\} \qquad \text{for } u, v \in X.$$

Then $G \leqq 2$ is a metric on X named after Gleason [4]. For $1 \in B \subset B(X, R)$ a real linear subspace which separates the points of X one defines

$$H: H(u, v) = \log \inf\{1 \leqq \lambda \in R \text{ with } (*)\colon F(u) \leqq \lambda F(v) \text{ and } F(v) \leqq \lambda F(u) \text{ for all } 0 \leqq F \in B\} \qquad \text{for } u, v \in X,$$

where $H(u, v) = \infty$ if no $1 \leqq \lambda \in R$ with (*) exists. Then $H \leqq \infty$ is a metric (in the extended sense) on X named after Harnack [1]. It admits the beautiful interpretation as the intrinsic metric for the state space $S(B) = \{\phi \in B^*\colon \ \phi(1) = \|\phi\| = 1\} \subset B^*$ restricted to $X \subset S(B)$ [2]. In the case $B = \operatorname{Re} A$ one has the famous result that $G(u, v) < 2 \Leftrightarrow \sigma(u, v) < 1 \Leftrightarrow H(u, v) < \infty$ for $u, v \in X$ [3], [1]. This is an equivalence relation on X which produces the Gleason parts for A. Furthermore G and H are equivalent metrics on X [1]. We claim the subsequent quantitative version of these facts. An essential part of the proof is contained in [5].

THEOREM. *In the case $B = \operatorname{Re} A$ we have*

$$H(u, v) = \log \frac{1 + \sigma(u, v)}{1 - \sigma(u, v)} = 2 \log \frac{2 + G(u, v)}{2 - G(u, v)} \qquad \text{for } u, v \in X.$$

PROOF. We can assume A to be norm closed. Let $u, v \in X$. From [5, Satz 1.8] we have $\sigma(u, v) = 4G(u, v)/(4 + (G(u, v))^2)$ or $(1 - \sigma(u, v))/(1 + \sigma(u, v)) = ((2 - G(u, v))/(2 + G(u, v)))^2$. Thus $\sigma(u, v) < 1 \Leftrightarrow G(u, v) < 2$. And from [5, Spezialfall 1.5] we have

$$((1 - \sigma(u, v))/(1 + \sigma(u, v)))F(u) \leqq F(v)$$

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for all $0 \leqq F \in \text{Re } A$. Thus if $\sigma(u, v) < 1$, then

$$\lambda = (1 + \sigma(u, v))/(1 - \sigma(u, v))$$

fulfills (*). For the converse we claim that if $1 \leqq \lambda \in R$ fulfills (*) then $\sigma(u, v) \leqq (\lambda - 1)/(\lambda + 1)$. It follows that $\sigma(u, v) < 1$ and $\lambda \geqq (1 + \sigma(u, v))/(1 - \sigma(u, v))$ so that the assertion will be proved. We take $f \in A$ with $\|f\| \leqq 1$ and $f(u) = 0$ and have to show that $|f(v)| \leqq (\lambda - 1)/(\lambda + 1)$. We can assume that $\|f\| < 1$ and $f(v) \geqq 0$. Then $P = (1 + f)/(1 - f) \in A$ with $\text{Re } P = (1 - |f|^2)/|1 - f|^2 \geqq 0$. Thus

$$\text{Re } P(v) = (1 + f(v))/(1 - f(v)) \leqq \lambda \text{ Re } P(u) = \lambda$$

or

$$f(v) \leqq (\lambda - 1)/(\lambda + 1).$$

The proof is complete.

REMARK ADDED IN PROOF: Because of [5, Korollar 1.3 and Korollar 1.6] the function σ is also a metric on X which in view of the above theorem is equivalent to H and G. The author is indebted to the referee for this remark.

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