A NOTE ON EXPANSIVE MAPPINGS

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Let f be a continuous multi-valued transformation of a metric space X (with metric d) onto itself. For brevity, call f a mapping.

If $x \in X$, an orbit of x under f is a set of the form

$$\{x_n: x_0 = x, x_{n+1} \in f(x_n) \text{ for each integer } n\}.$$

We say that f is expansive on X if there exists $\delta > 0$ such that $x, y \in X$, $x \neq y$ implies that for each orbit A of x and for each orbit B of y, there exist $x_n \in A$, $y_n \in B$ such that $d(x_n, y_n) > \delta$. (See [2].) The quantity δ is called an expansive constant for f. This generalizes the concept of expansive homeomorphism studied in [1], for instance.

It is known that if f is a homeomorphism of $[0, 1]$ onto itself, then f is not expansive. (See [1].) The purpose of this paper is to show that f is not expansive if it is a single-valued mapping $[0, 1]$ onto itself, but that there are expansive mappings on $[0, 1]$.

To show this, we need some preliminary results.

THEOREM 1. *If f is a single-valued, uniformly continuous mapping of X onto itself, then f is expansive if and only if f^n is expansive for $n \neq 0$.*

PROOF. Since it is obvious that f is expansive if and only if f^{-1} is, let us assume that $n > 0$. It is also clear that if f^n is expansive, then f is. Assume therefore that f is expansive with expansive constant δ. By uniform continuity, there exists $\Delta > 0$ such that $d(x, y) < \Delta$ implies $d(f^i(x), f^i(y)) < \delta$ for $i = 0, 1, \cdots, n-1$. Suppose that $\Delta/2$ is not an expansive constant for f^n. Then there exist distinct points x and y, an orbit A of x under f^n and an orbit B of y under f^n, such that for each integer k, $x_k \in A$ and $y_k \in B$ implies $d(x_k, y_k) \leqq \Delta/2$. Let m be any integer. Then there exist integers i and j such that $m = nj + i$, where $0 \leqq i \leqq n-1$. Define $x_m = f^i(x_{nj})$ and $y_m = f^i(y_{nj})$. Since $f^i(f^{nj}) = f^m$, the x_m's and y_m's define orbits under f of x and y respectively. Also, $d(x_m, y_m) < \delta$ for each m, contradicting the assumption that f is expansive with expansive constant δ.

LEMMA. *If f is a single-valued mapping of $[0, 1]$ onto itself, then f^2 has at least two fixed points.*

Received by the editors August 21, 1968.

PROOF. Certainly f has at least one fixed point x_0. We may assume that $f(x)>x$ for $0\leqq x<x_0$ and $f(x)<x$ for $x_0<x\leqq 1$, for otherwise f and hence f^2 will have at least two fixed points. Hence, by onto-ness, there exists $x_1\in[0, x_0)$ such that $f(x_1)=1$ and there exists $x_2\in(x_0, 1]$ such that $f(x_2)=0$. By the intermediate value theorem, there exists $x_3\in(x_0, x_2]$ such that $f(x_3)=x_1$. Hence $f^2(x_3)=1\geqq x_3$, and f^2 has a fixed point distinct from x_0.

THEOREM 2. *If f is a single-valued mapping of* $[0, 1]$ *onto itself, then f is not expansive.*

PROOF. Suppose that f is expansive with expansive constant δ. Using Theorem 1 and the preceding lemma, we may assume that f has at least two fixed points. It is clear that f can have only a finite number of fixed points, so let a and b be fixed with $b>a$, and let there be no fixed points between a and b.

Case 1. Let $f(x)>x$ for $a<x<b$. If f is monotone increasing on $[a, b]$, then for each $x\in(a, b)$, there exists an orbit A of x such that $x_n\in A$ implies $\lim_{n\to-\infty} x_n=a$ and $\lim_{n\to\infty} x_n=b$. It is therefore clear that there exist $x, y\in(a, b)$ with orbits A and B respectively such that for each integer n, $x_n\in A$, $y_n\in B$ implies $d(x_n, y_n)<\delta$. (This is essentially the proof of the nonexistence of an expansive homeomorphism on $[0, 1]$ given in [1].)

Suppose there exists $c\in(a, b)$ such that f is monotone increasing on $[a, c]$, and that c is the smallest such number. Then there exist distinct points x and y, arbitrarily near c, such that $f(x)=f(y)$, and there exist orbits A of x and B of y such that $x_n\in A$, $y_n\in B$, $n<0$, implies $d(x_n, y_n)<\delta$. (Again, f is a homeomorphism on $[a, c]$, and $\lim_{n\to-\infty} x_n =\lim_{n\to-\infty} y_n=a$.)

Finally, suppose that f is not monotone on any interval $[a, c]$, where $a<c<b$. Then, by repeated use of the intermediate value theorem, there exist distinct x and y in $[a, a+\delta]$, with orbits A and B respectively such that $f(x)=f(y)$, and $x_n\in A$, $y_n\in B$, $n<0$ implies $x_n\in[a, x]$, $y_n\in[a, y]$. Thus $d(x_n, y_n)\leqq\delta$ for each n, again contradicting the expansiveness of f.

Case 2. Let $f(x)<x$ for $a<x<b$. Considering intervals of the form $[c, b]$, $a<c<b$, the analysis of Case 1 can essentially be repeated, and the theorem follows.

The above theorem is not valid if one does not assume that f is single-valued. Consider the following example:

Let g be defined on $[0, 1]$ by $g(x)=e^{2\pi ix}$, and let h be defined on the unit circle by $h(z)=z^2$. Let f be the (multi-valued) mapping of $[0, 1]$ onto itself defined by $f(x)=g^{-1}hg(x)$. It is clear that $f^n(x)=g^{-1}h^ng(x)$

for $n=1, 2, \cdots$, and it is also clear that the positive iterates of h spread out distinct points on the unit circle. It follows that f is an expansive mapping on $[0, 1]$.

Bibliography

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