ON THE ZEROS OF THE RIEMANN ZETA-FUNCTION

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1. Introduction. If

$$R(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

the functional equation for the Riemann zeta-function $\zeta(s)$ is given by $R(s)=R(1-s)$. Berlowitz [1] has recently shown that if $0<\lambda<1$, then both Re $R(\lambda+it)$ and Im $R(\lambda+it)$ vanish infinitely often. In this note we shall give an improvement on this result.

Let $N_R(\lambda, T)$ denote the number of zeros for Re $R(\lambda+it)$ on $0<t<T$. Similarly, $N_I(\lambda, T)$ denotes the number of zeros for Im $R(\lambda+it)$ on $0<t<T$.

THEOREM. *If* $0<\lambda<1$, *then*

$$N_R(\lambda, T) > AT \tag{1.1}$$

and

$$N_I(\lambda, T) > AT, \tag{1.2}$$

where $A=A(\lambda)$.

Here and elsewhere A, A_1 and A_2 denote positive constants and K_1 and K_2 complex constants, none of which is necessarily the same with each occurrence.

For $\lambda=\frac{1}{2}$, the result follows from a famous theorem of Hardy and Littlewood [2, p. 222]. In fact, we use their method [2, pp. 222–226] in our proof.

2. Proof of the theorem. We prove (1.1) for the case $0<\lambda<\frac{1}{2}$. The proof for $\frac{1}{2}<\lambda<1$ will then follow from the functional equation $R(s)=R(1-s)$.

From [1] we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{Re} R(\lambda + it)e^{i\xi t}dt = (e^{-\lambda\xi} + e^{-(1-\lambda)\xi})\psi(e^{-2\xi}) - \tfrac{1}{2}(e^{\lambda\xi} + e^{(1-\lambda)\xi}),$$

where $\psi(x) = \sum_{n=1}^{\infty} \exp(-n^2\pi x)$ and Re $e^{-2\xi}>0$. Putting

$$\xi = -i(\pi/4 - \delta/2) - y, \qquad \delta > 0,$$

we see that

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$$F(t) = \frac{1}{(2\pi)^{1/2}} \operatorname{Re} R(\lambda + it) e^{(\pi/4-\delta/2)t}$$

and

$$\begin{aligned} f(y) = {} & (e^{\lambda\{i(\pi/4-\delta/2)+y\}} + e^{(1-\lambda)\{i(\pi/4-\delta/2)+y\}})\psi(e^{i(\pi/2-\delta)+2y}) \\ & - \tfrac{1}{2}(e^{-\lambda\{i(\pi/4-\delta/2)+y\}} + e^{-(1-\lambda)\{i(\pi/4-\delta/2)+y\}}) \end{aligned}$$

are Fourier transforms. We now use formula (10.7.1), [2, p. 223], i.e.

$$\tag{2.1} \begin{aligned} \int_{-\infty}^{\infty} \left| \int_t^{t+H} F(u)du \right|^2 dt \leqq 2H^2 \int_0^{1/H} |f(y)|^2 dy \\ + 8 \int_{1/H}^{\infty} |f(y)|^2 y^{-2} dy, \end{aligned}$$

where $H \geqq 1$ is a constant to be chosen later. Letting $y = \log x$ and $G = e^{1/H}$, we see that (2.1) yields

$$\tag{2.2} \begin{aligned} & \int_{-\infty}^{\infty} \left| \int_t^{t+H} F(u)du \right|^2 dt \\ & \qquad = O\left\{ H^2 \int_1^G |\psi(e^{i(\pi/2-\delta)}x^2)|^2 dx \right\} \\ & \qquad + O\left\{ \int_G^{\infty} |\psi(e^{i(\pi/2-\delta)}x^2)|^2 \frac{x^{1-2\lambda}}{\log^2 x} dx \right\} + O(H). \end{aligned}$$

The first integral on the right side of (2.2) is estimated in [2, p. 224], and is $O(H\delta^{-1/2})$ as δ tends to 0.

To estimate the second we write

$$\tag{2.3} \begin{aligned} & |\psi(e^{i(\pi/2-\delta)}x^2)|^2 \\ & \quad = \sum_{n=1}^{\infty} \exp[-2n^2\pi x^2 \sin\delta] \\ & \quad + \sum_{\substack{m=1 \\ m\neq n}}^{\infty} \sum_{n=1}^{\infty} \exp[-(m^2+n^2)\pi x^2 \sin\delta + i(m^2-n^2)\pi x^2 \cos\delta]. \end{aligned}$$

To examine the contribution of the first sum on the right side of (2.3), divide the interval (G, ∞) into the subintervals $(G, \delta^{-1/2})$ and $(\delta^{-1/2}, \infty)$ and let the corresponding integrals be denoted by I_1 and I_2, respectively. The sum under consideration is $O(x^{-1}\delta^{-1/2})$, and so

$$
\begin{aligned}
I_1 &= O\left(\delta^{-1/2}\int_G^{\delta^{-1/2}} \frac{x^{1-2\lambda}}{x\log^2 x}\,dx\right)\\
&= O(H\delta^{\lambda-1}),
\end{aligned}
$$

upon an integration by parts. The sum is also $O(\exp(-2\pi x^2\delta))$ for $x^2\delta \geqq 1$, and so

$$
\begin{aligned}
I_2 &= O\left(\int_{\delta^{-1/2}}^{\infty} \frac{x^{1-2\lambda}}{\log^2 x}\exp(-2\pi x^2\delta)dx\right)\\
&= O\left(\delta^{\lambda}\int_{\delta^{-1/2}}^{\infty} x\exp(-2\pi x^2\delta)dx\right)\\
&= O(\delta^{\lambda-1}).
\end{aligned}
$$

The contribution of the second term on the right side of (2.3) is $O(H\delta^{-1/2})$ by the same argument as that in [2, p. 224].

Letting $\delta = 1/T$ and

$$
I = \int_t^{t+H} \operatorname{Re} R(\lambda + iu)e^{(\pi/4-1/T)u}du,
$$

we have shown that

$$
\int_T^{2T} |I|^2 dt = O(HT^{1-\lambda}). \tag{2.4}
$$

From the functional equation for $\zeta(s)$,

$$
\begin{aligned}
\operatorname{Re} R(\lambda + it) &= \tfrac{1}{2}\{R(\lambda + it) + R(\lambda - it)\}\\
&= \tfrac{1}{2}\{R(\lambda + it) + R(1 - \lambda + it)\}.
\end{aligned}
$$

Using the above and Stirling's formula for $\Gamma(\sigma+it)$, we find that

$$
|\operatorname{Re} R(\lambda + it)| \geqq e^{-\pi t/4}|K_1 t^{(\lambda-1)/2}\zeta(\lambda + it) + K_2 t^{-\lambda/2}\zeta(1 - \lambda + it)|.
$$

Thus, if $T \leqq t \leqq 2T$,

$$
\begin{aligned}
J &= \int_t^{t+H} |\operatorname{Re} R(\lambda + iu)|\, e^{(\pi/4-1/T)u}du\\
&\geqq T^{(\lambda-1)/2}\int_t^{t+H} |K_1\zeta(\lambda + iu) + K_2 u^{1/2-\lambda}\zeta(1 - \lambda + iu)|\, du\\
&\geqq T^{(\lambda-1)/2}\left|\int_t^{t+H} \{K_1\zeta(\lambda + iu) + K_2 u^{1/2-\lambda}\zeta(1 - \lambda + iu)\}du\right|.
\end{aligned}
$$

Using a simple approximation for $\zeta(s)$ [2, p. 67], we have

$$T^{(1-\lambda)/2}J \geqq \left| \int_t^{t+H} \left\{ K_1\left(\sum_{n \leqq AT} n^{-\lambda-iu} + O(T^{-\lambda}) \right) \right.\right.$$

$$\left.\left. + K_2 u^{1/2-\lambda}\left(\sum_{n \leqq AT} n^{\lambda-1-iu} + O(T^{\lambda-1}) \right) \right\} du \right|$$

$$\geqq (A_1 + A_2 T^{1/2-\lambda})H + O\left\{ \left| \int_t^{t+H} \sum_{2 \leqq n \leqq AT} n^{-\lambda-iu} du \right| \right\}$$

$$+ O\left\{ \left| \int_t^{t+H} u^{1/2-\lambda} \sum_{2 \leqq n \leqq AT} n^{\lambda-1-iu} du \right| \right\} + O(HT^{-\lambda}).$$

Employing the second mean value theorem for integrals to the second integral on the right side, we have

$$T^{(1-\lambda)/2}J \geqq A_1 T^{1/2-\lambda} H + O(HT^{-\lambda})$$

$$+ O\left\{ \left| \sum_{2 \leqq n \leqq AT} (1/n^{\lambda+i(t+H)} \log n - 1/n^{\lambda+it} \log n) \right| \right\} \tag{2.5}$$

$$+ O\left\{ T^{1/2-\lambda} \left| \sum_{2 \leqq n \leqq AT} (1/n^{1-\lambda+i(t+H)} \log n - 1/n^{1-\lambda+i\tau} \log n) \right| \right\}$$

$$\geqq A_1 T^{1/2-\lambda} H + \Psi,$$

say, where $t<\tau<t+H$.

We show next that

$$\int_T^{2T} |\Psi|^2 dt = O(T^{2-2\lambda}). \tag{2.6}$$

Clearly, it is sufficient to examine

$$\int_T^{2T} \left| \sum_{2 \leqq n \leqq AT} 1/n^{\lambda+it} \log n \right|^2 dt$$

$$= \int_T^{2T} \sum_{2 \leqq m \leqq AT} 1/m^{\lambda+it} \log m \sum_{2 \leqq n \leqq AT} 1/n^{\lambda-it} \log n \, dt$$

$$= T \sum_{2 \leqq n \leqq AT} 1/n^{2\lambda} \log^2 n + \sum_{\substack{2 \leqq m,n \leqq AT \\ m \neq n}}\sum (1/(mn)^\lambda \log m \log n) \int_T^{2T} (n/m)^{it} dt$$

$$= O(T^{2-2\lambda}) + O\left(\sum_{2 \leqq m < n \leqq AT}\sum 1/(mn)^\lambda \log m \log n \log n/m \right).$$

This last sum is estimated in exactly the same manner as (7.2.1), [2, p. 116], and is $O(T^{2-2\lambda})$. Replacing λ by $1-\lambda$ in the above calculation, we find

$$\int_T^{2T}\left|\sum_{2\leqq n\leqq AT} 1/n^{1-\lambda+it}\right|^2 dt = O(T^{2\lambda}).$$

Thus, we have proved (2.6).

Now let S denote the subset of $(T, 2T)$ where $I=J$. Thus,

$$\int_S |I|\,dt = \int_S J dt.$$

From (2.4),

$$\int_S |I|\,dt \leqq \int_T^{2T} |I|\,dt \leqq \left\{T\int_T^{2T} |I|^2 dt\right\}^{1/2} \tag{2.7}$$
$$\leqq A_2 H^{1/2} T^{1-\lambda/2}.$$

Also, from (2.5) and (2.6),

$$\begin{aligned}\int_S J dt &\geqq T^{(\lambda-1)/2}\int_S (A_1 T^{1/2-\lambda} H + \Psi) dt \\ &\geqq A_1 H T^{-\lambda/2} m(S) - T^{(\lambda-1)/2}\int_T^{2T} |\Psi|\,dt \\ &\geqq A_1 H T^{-\lambda/2} m(S) - T^{(\lambda-1)/2}\left\{T\int_T^{2T} |\Psi|^2 dt\right\}^{1/2} \\ &\geqq A_1 H T^{-\lambda/2} m(S) - A T^{1-\lambda/2},\end{aligned}$$

where $m(S)$ denotes the measure of S. Combining the above with (2.7), we find that

$$A_1 H T^{-\lambda/2} m(S) \leqq A T^{1-\lambda/2} + A_2 H^{1/2} T^{1-\lambda/2},$$

or

$$m(S) \leqq A H^{-1/2} T.$$

Divide $(T, 2T)$ into $[T/2H]$ pairs of abutting intervals j_1, j_2, each, except for possibly the last j_2, of length H. Then, if j_1 does not contain entirely points of S, either j_1 or j_2 contains a zero of $\operatorname{Re} R(\lambda+it)$. Thus, if there are ν j_1-intervals containing only points of S, $\nu H \leqq m(S)$, and $\operatorname{Re} R(\lambda+it)$ has at least

$$[T/2H] - \nu > T/3H - AT/H^{3/2} > T/4H$$

zeros if H is large enough.

To prove (1.2) we merely observe that

$$i \operatorname{Im} R(\lambda + it) = \tfrac{1}{2}\{R(\lambda + it) - R(1 - \lambda + it)\},$$

and then proceed as before.

References

1. B. Berlowitz, *Extensions of a theorem of Hardy*, Acta Arith. **14** (1968), 203–207.

2. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Clarendon Press, Oxford, 1951.

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