GALOIS COHOMOLOGY AND CLASS NUMBER IN CONSTANT EXTENSION OF ALGEBRAIC FUNCTION FIELDS

JAMES R. C. LEITZEL

Let F be a field of algebraic functions of one variable having the finite field K as exact field of constants. The class number of F is defined as the order of the finite group, $C_0(F)$, of divisor classes of degree zero. Let L be the unique cyclic extension of K of degree n, $E=F\cdot L$ the corresponding constant extension with galois group G. Since K is perfect, the canonical homomorphism of the group of divisor classes of F in the group of divisor classes of E is an injection [2, p. 477]. If h_E, h_F denote the class numbers of E and F, respectively, we have $h_E=h_F\cdot k$, for some integer k. The purpose of this note is to prove the following two theorems:

THEOREM 1. *If E/F is a constant extension of the algebraic function field F and if G is the corresponding galois group, then $H^i(G, C_0(E))=0$ for all i.*

THEOREM 2. *If E/F is a constant extension of prime degree p, then $h_E=h_F\cdot k$ where $k\equiv 1 \bmod p$ if $p\nmid h_F$ and $k\equiv 0 \bmod p^t$ if $p\mid h_F$ and t is the p-rank of $C_0(F)$.*

Throughout this note E/F will denote a cyclic constant extension of the algebraic function field F with galois group G generated by σ. In a natural fashion G operates on the prime divisors of E. Thus if $N_{E/F}=1+\sigma+\cdots+\sigma^{n-1}$, $n=[E: F]$, we have for a prime $\mathfrak{P}$ of E, that $N_{E/F}\mathfrak{P}=\mathfrak{p}^{n(\mathfrak{P})}$ where $\mathfrak{p}=rst_F\mathfrak{P}$ and $n(\mathfrak{P})=[E(\mathfrak{P}): F(\mathfrak{p})]$ the degree of the corresponding completions since the extension is everywhere unramified. We see easily then that $\deg_{F/K}N_{E/F}(\mathfrak{P})=n\deg_{E/L}\mathfrak{P}$. This norm map extends from the prime divisors to the full divisor group $D(E)$ in the natural way and it is compatible with the field norm and formation of principal divisors: that is, if we use parentheses to denote principal divisors and $\alpha\in E$, then $N_{E/F}(\alpha)=(N_{E/F}\alpha)$. We shall write the group operation in $D(E)$ additively and denote as usual the subgroup of principal divisors by $P(E)$ and the divisors of degree zero by $D_0(E)$. Similar notations will be used for the field F.

I. **Proof of Theorem 1.** In the normal extension E/F all primes $\mathfrak{P}$ of E extending a fixed prime $\mathfrak{p}$ of F are conjugate. Thus for $\mathfrak{a}\in D_0(E)$ we have $\mathfrak{a}^\sigma=\mathfrak{a}$ for all $\sigma\in G$ if and only if $\mathfrak{a}\in D_0(F)$; therefore $D_0(E)^G$

Received by the editors October 8, 1968.

$=D_0(F)$. From the exact G-sequence $0\to L\to E\to P(E)\to 0$ we get the exact sequence $0\to L^G\to E^G\to P(E)^G\to H^1(G, L)\to H^1(G, E)\to H^1(G, P(E))\to H^2(G, L)\to \cdots$. Using Hilbert's Theorem 90 and the fact that L is a finite field, we conclude

$$(1)\qquad P(E)^G = P(F) \quad \text{and} \quad H^1(G, P(E)) = 0.$$

Consequently from the exact G-sequence $0\to P(E)\to D_0(E)\to C_0(E)\to 0$ and (1) we derive

$$(2)\qquad C_0(E)^G = C_0(F).$$

We next claim that the induced map $N_{E/F}$: $C_0(E)\to C_0(F)$ is surjective. Let $J(E)$ denote the idèle group of E and define ϕ: $J(E)\to D(E)$ by $\phi(A)=\sum v_{\mathfrak{P}}(A)\mathfrak{P}$ where $v_{\mathfrak{P}}(A)=v_{\mathfrak{P}}(A_{\mathfrak{P}})$, $A_{\mathfrak{P}}$ the $\mathfrak{P}$ component of the idèle A. $\phi(A)$ is a divisor since A is a unit almost everywhere. It is easily checked that ϕ is surjective. Let $J(E)^0=\phi^{-1}(D_0(E))$. Recall that $J(E)$ is also a G-module and the norm on idèles is compatible with the norm on divisors [4]. Thus we have the exact and commutative diagram:

$$(3)\qquad \begin{array}{ccc} J(E)^0 & \xrightarrow{\phi} & D_0(E) \to 0 \\ N_{E/F}\downarrow & & \downarrow N_{E/F} \\ J(F)^0 & \xrightarrow{\phi} & D_0(F) \to 0 \end{array}$$

If $I(E)$ denotes the idèle class group of E, then class field theory [1, p. 79] asserts that $J(F)^0/F\subset N_{E/F}(I(E))$. Now suppose $a\in C_0(F)$ and $\mathfrak{a}\in D_0(F)$ is a representative for a. Let $A\in J(F)^0$ be such that $\phi(A)=\mathfrak{a}$. Then there exists $B\in J(E)^0$ such that $N_{E/F}B=A(\beta)$, $\beta\in F$. If b is the class of $\phi(B)$ in $C_0(E)$, we conclude from (3) that $N_{E/F}b=a$. Hence $N_{E/F}$ is surjective.

We have therefore proved $H^0(G, C_0(E))=0$. But since $C_0(E)$ is a finite group the Herbrand quotient gives that $H^1(G, C_0(E))=0$ as well. Using that G is cyclic and consequently has periodic cohomology, we have proved Theorem 1.

II. **Proof of Theorem 2.** Since $C_0(E)^G=C_0(F)$ we can induce a G-action on the factor group $C_0(E)/C_0(F)$ and get the exact G-sequence

$$0 \to C_0(F) \to C_0(E) \to C_0(E)/C_0(F) \to 0.$$

From this we derive

$$0 \to C_0(F)^G \to C_0(E)^G \to (C_0(E)/C_0(F))^G \to H^1(G, C_0(F)) \to H^1(G, C_0(E)) \to \cdots.$$

Using Theorem 1 and the trivial action of G on $C_0(F)$ we see that

$$(C_0(E)/C_0(F))^G \cong H^1(G, C_0(F)). \tag{4}$$

Furthermore in the case of trivial action we have [**3**, p. 142]

$$|H^1(G, C_0(F))| = p^t, \quad \text{where } t \text{ is the } p\text{-rank of } C_0(F). \tag{5}$$

Therefore if $p|h_F$, we have immediately that for $k = |C_0(E)/C_0(F)|$, $k \equiv 0 \bmod p^t$, since $(C_0(E)/C_0(F))^G$ is a subgroup of $C_0(E)/C_0(F)$.

On the other hand, taking the decomposition of $C_0(E)/C_0(F)$ into G orbits we see that $k = \sum [G: H_c]$ where the summation extends over a set of representatives for the various orbits and H_c is the corresponding stabilizer. Therefore if G is of prime order p, we have $[G: H_c] = 1$ or p and $[G: H_c] = 1$ if and only if $G = H_c$. Tracing the action of G on $C_0(E)/C_0(F)$, we see that this is the case if and only if $c \in (C_0(E)/C_0(F))^G$. Therefore we have

$$k = |(C_0(E)/C_0(F))^G| + sp. \tag{6}$$

Hence if $p \nmid h_F$ from (4), (5) and (6) we conclude that $k \equiv 1 \bmod p$. The following remarks are immediate consequences of Theorem 2.

1. If F is a function field in one variable with finite field of constants k and p is a prime with $p^\alpha \| h_F$, $\alpha \geqq 1$, then there is a constant extension E/F with $p^{\alpha+1} | h_E$.

2. If E/L is a constant extension of the algebraic function field F/K of prime degree p then $p | h_E$ if and only if $p | h_F$ $(h_F \neq 1)$.

Bibliography

1. E. Artin and J. Tate, *Class field theory*, Seminar notes, Institute for Advanced Study, Princeton, N. J., 1960.

2. H. Hasse, *Zahlentheorie*, Akademie-Verlag, Berlin, 1963.

3. J. P. Seer, *Corps locaux*, Hermann, Paris, 1962.

4. G. Whaples, *Class field theory*, Lecture notes, Indiana Univ., Bloomington, 1963.

Ohio State University