ON THE RELATION BETWEEN THE ABEL AND BOREL-TYPE METHODS OF SUMMABILITY

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1. **Introduction.** It is known that the Abel method and the Borel exponential method of summability are not equivalent, but that under certain conditions, both methods sum the same series to the same sum [5]. This was recently extended in one direction, to the conditions under which a series summable by a Borel-type method is also summable by the Abel method [7]. The object of this paper is to extend this last result to absolute summability.

2. **Definitions and generalities.** Suppose throughout that σ, a_n ($n=0, 1, \cdots$) are arbitrary complex numbers, that $\alpha>0$ and that β is real. Let N be any nonnegative integer greater than $1-\beta/\alpha$. Let M denote a positive constant, not necessarily the same at each occurrence.

Define

$$s_n = \sum_{r=0}^{n} a_r; \qquad s_{-1} = 0; \qquad \sigma_N = \sigma - s_{N-1}.$$

2.1. *Definitions of the Borel-type methods of summability.*
Define

$$a(x) = \sum_{n=N}^{\infty} \frac{a_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n + \beta)}; \qquad s(x) = \sum_{n=N}^{\infty} \frac{s_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n + \beta)}.$$

It is known [1] that the convergence of one of these series for all $x \geqq 0$ implies the convergence of the other for all $x \geqq 0$; henceforth it is assumed that these series are convergent for all $x \geqq 0$.

Define $S(x) = S_{\alpha,\beta}(x) = \alpha e^{-x} s(x)$; $A(x) = \int_0^x e^{-t} a(t)dt$.

Note. Except in the lemma in §4, the suffixed form $S_{\alpha,\beta}(x)$ will not be used.

ORDINARY SUMMABILITY [2]. If $S(x) \to \sigma$ as $x \to \infty$, then $s_n \to \sigma(B, \alpha, \beta)$. If $A(x) \to \sigma_N$ as $x \to \infty$, then $s_n \to \sigma(B', \alpha, \beta)$.

ABSOLUTE SUMMABILITY [4]. If $s_n \to \sigma(B, \alpha, \beta)$ and $S(x)$ is of bounded variation with respect to x on the interval $[0, \infty)$ then $s_n \to \sigma|B, \alpha, \beta|$. If $s_n \to \sigma(B', \alpha, \beta)$ and $A(x)$ is of bounded variation with respect to x on the interval $[0, \infty)$, then $s_n \to \sigma|B', \alpha, \beta|$.

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Note. The summability method $(B, 1, 1)$ is the classical Borel exponential method (B) and the method $(B', 1, 1)$, the classical Borel integral method (B').

The actual choice of N is immaterial. Thus N will henceforth be assumed to be sufficiently large so that $S(0)=0$ and $x^{1-\beta}S'(x)$ is continuous for $x \geqq 0$. Further, it may be assumed without loss of generality that $a_0 = a_1 = \cdots = a_{N-1} = 0$, so that $\sigma_N = \sigma$.

2.2. *Definitions of the Abel methods of summability.* Define

$$L(x) = \sum_{n=0}^{\infty} a_n x^n = (1 - x) \sum_{n=0}^{\infty} s_n x^n.$$

ORDINARY SUMMABILITY [6, p. 7]. If $L(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent for $|x| < 1$ and $L(x) \to \sigma$ as $x \to 1-$, then $s_n \to \sigma(A)$.

ABSOLUTE SUMMABILITY [8]. If $s_n \to \sigma(A)$ and $L(x)$ is of bounded variation with respect to x on the interval $[0, 1)$, then $s_n \to \sigma|A|$.

3. **Theorems for ordinary methods.** In 1931, Doetsch [5] proved the following theorem:

THEOREM A. *If $s_n \to \sigma(B)$ and $L(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent for $|x| < 1$, then $s_n \to \sigma(A)$.*

This was extended in 1961 by Jajte [7] to give

THEOREM B. *If $s_n \to \sigma(C, k)(B, \alpha, \beta)$ where $0 \leqq k \leqq 1$, and $L(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent for $|x| < 1$, then $s_n \to \sigma(A)$.*

Note. $(C, k)(B, \alpha, \beta)$ is the (B, α, β) method applied to the (C, k) mean of s_n (the Cesaro mean of order k [6, p. 96]).

Since $(C, 0)$ is convergence, the relation between the Borel-type method and the Abel methods is expressed as

COROLLARY B. *If $s_n \to \sigma(B, \alpha, \beta)$ and $L(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent for $|x| < 1$, then $s_n \to \sigma(A)$.*

Since it is known [3, Theorem 2] that $s_n \to \sigma(B, \alpha, \beta)$ if and only if $s_n \to \sigma(B', \alpha, \beta - 1)$, the following theorem is immediate:

THEOREM 1. *If $s_n \to \sigma(B', \alpha, \beta)$ and $L(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent for $|x| < 1$, then $s_n \to \sigma(A)$.*

4. **Theorems for absolute methods.** In this section, Corollary B and Theorem 1 are extended to absolute summability.

THEOREM 2. *If $s_n \to \sigma|B, \alpha, \beta|$ and $L(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent for $|x| < 1$, then $s_n \to \sigma|A|$.*

PROOF. Because of Corollary B and since $S(x)$, $L(x)$ are absolutely continuous on $[0, \infty)$, $[0, 1)$ respectively, it is sufficient for the proof of Theorem 2 to prove that

$$\int_0^1 |L'(t)|\,dt < \infty \quad \text{whenever} \quad \int_0^\infty |S'(t)|\,dt < \infty.$$

The following lemma is required:

LEMMA. *If* $s_n \to \sigma |B, \alpha, \beta|$ *then* $s_n \to \sigma |B, \alpha, \beta+\delta|$ *whenever* $\delta > 0$.

PROOF. (*Note.* In this proof, the suffixed form $S_{\alpha,\beta}(x)$ is used.) Since it is known that $s_n \to \sigma(B, \alpha, \beta+\delta)$ whenever $s_n \to \sigma(B, \alpha, \beta)$ and $\delta > 0$ [**3**, Result II], it suffices to show that

$$\int_0^\infty |S'_{\alpha,\beta+\delta}(t)|\,dt < \infty \quad \text{whenever} \quad \int_0^\infty |S'_{\alpha,\beta}(t)|\,dt < \infty.$$

Thus, since [**4**, Result I]

$$\Gamma(\delta) S'_{\alpha,\beta+\delta}(t) = e^{-t}\int_0^t (t-u)^{\delta-1} e^u S'_{\alpha,\beta}(u)\,du,$$

it follows that

$$\begin{aligned}
\Gamma(\delta)\int_0^\infty |S'_{\alpha,\beta+\delta}(t)|\,dt &\leqq \int_0^\infty e^{-t}dt \int_0^t (t-u)^{\delta-1} e^u |S'_{\alpha,\beta}(u)|\,du \\
&= \int_0^\infty e^u |S'_{\alpha,\beta}(u)|\,du \int_u^\infty (t-u)^{\delta-1} e^{-t}dt \\
&= \Gamma(\delta)\int_0^\infty |S'_{\alpha,\beta}(u)|\,du < \infty.
\end{aligned}$$

This completes the proof of the lemma.

The direct proof of Corollary B consists of taking the Laplace transform of $S(x)$ and knowing that whenever $S(x) \to \sigma$ as $x \to \infty$,

$$I(y) = y\int_0^\infty e^{-yu} S(u)\,du \to \sigma \quad \text{as } y \to 0+,$$

$$B(y) = (1+y)^{\beta-\alpha}\left\{\frac{(1+y)^\alpha - 1}{\alpha y}\right\} \to 1 \quad \text{as } y \to 0+,$$

and $L(x) = B(y)I(y)$ where x and y are related by $x = (1+y)^{-\alpha}$.

Note. This relation between x and y is assumed implicitly for the remainder of this proof.

First, note that

$$I(y) = \int_0^\infty e^{-yu} S'(u)du \quad \text{and} \quad I'(y) = -\int_0^\infty e^{-yu} uS'(u)du.$$

In order to show that $L(x)$ is of bounded variation with respect to x on the interval $[0, 1)$, it is sufficient to prove that

$$\int_0^\infty \left| \frac{d}{dy} B(y)I(y) \right| dy < \infty.$$

Now, note the following properties of $B(y)$:

(i) $B(y) \to 1$ as $y \to 0+$.

(ii) $B(y)$ is continuous for $y > 0$.

(iii) $B(y) \sim y^{\beta-1}/\alpha$ as $y \to \infty$.

Also, for $y > 0$

$$\begin{aligned}\frac{B'(y)}{B(y)} &= \frac{\beta-\alpha}{1+y} - \frac{1}{y} + \frac{\alpha(1+y)^{\alpha-1}}{\{(1+y)^\alpha - 1\}} \\ &= \frac{\beta-\alpha}{1+y} + \frac{\alpha y(1+y)^{\alpha-1} - (1+y)^\alpha + 1}{y\{(1+y)^\alpha - 1\}}.\end{aligned}$$

Thus, $B'(y)$ has the following properties:

(iv) $B'(y) \to (2\beta - \alpha - 1)/2$ as $y \to 0+$.

(v) $B'(y)$ is continuous for $y > 0$.

(vi) $B'(y) \sim (\beta-1)y^{\beta-2}/\alpha$ as $y \to \infty$.

In view of all these properties, since $\beta > 1$ and since $t^{1-\beta}S'(t)$ is continuous for $t \geqq 0$, it now follows that

(a)
$$\begin{aligned}\int_1^\infty |B'(y)I(y)|\, dy &\leqq \int_1^\infty My^{\beta-2}dy \int_0^\infty e^{-yu} |S'(u)|\, du \\ &= M\int_0^\infty |S'(u)|\, du \int_1^\infty y^{\beta-2}e^{-yu}dy \\ &= M\int_0^\infty u^{1-\beta} |S'(u)|\, du < \infty,\end{aligned}$$

(b)
$$\begin{aligned}\int_1^\infty |B(y)I'(y)|\, dy &\leqq \int_1^\infty My^{\beta-1}dy \int_0^\infty ue^{-yu} |S'(u)|\, du \\ &= M\int_0^\infty u |S'(u)|\, du \int_1^\infty y^{\beta-1}e^{-yu}dy \\ &= M\int_0^\infty u^{1-\beta} |S'(u)|\, du < \infty,\end{aligned}$$

$$\text{(c)} \qquad \int_0^1 |B(y)I'(y)|\,dy \leqq \int_0^1 M\,dy \int_0^\infty ue^{-yu}|S'(u)|\,du$$
$$= M\int_0^\infty u|S'(u)|\,du \int_0^1 e^{-yu}dy$$
$$= M\int_0^\infty (1-e^{-u})|S'(u)|\,du < \infty,$$

and

$$\text{(d)} \qquad \int_0^1 |B'(y)I(y)|\,dy < \infty$$

since $B'(y)$ and $I(y)$ are bounded on $[0, 1]$.

Thus, it follows from (a), (b), (c) and (d), that

$$\int_0^\infty \left|\frac{d}{dy} B(y)I(y)\right| dy < \infty,$$

and this completes the proof of Theorem 2.

Since it is known that $s_n \to \sigma|B, \alpha, \beta|$ if and only if $s_n \to \sigma|B', \alpha, \beta-1|$ [4, Theorem 17], the following theorem follows immediately:

THEOREM 3. *If* $s_n \to \sigma|B', \alpha, \beta|$ *and* $L(x) = \sum_{n=0}^\infty a_n x^n$ *is convergent for* $|x| < 1$, *then* $s_n \to \sigma|A|$.

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