BASIS PRESERVING MAPS[1]

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The Paley-Wiener and Krein-Milman-Rutman theorems state that a sequence (y_i) in a B-space X which is sufficiently near a basic sequence (x_i) is itself a basic sequence, and that the linear extension of the map T defined by $Tx_i = y_i$ is an isomorphism of $[x_i]$ onto $[y_i]$ (see, e.g. [1], [6]). Here we examine some criteria for linear and nonlinear maps to preserve bases.

A sequence $(x_i) \subset X$ is *equivalent* to a sequence $(y_i) \subset Y$ if, for any sequence of scalars (a_i), $\sum a_i x_i$ converges if and only if $\sum a_i y_i$ converges [2]. A sequence (y_i) is equivalent to a basic sequence (x_i) if and only if there is an isomorphism $T:[x_i] \to [y_i]$ ($[x_i]$ = closed span (x_i), isomorphism = linear homeomorphism) such that $Tx_i = y_i$. As a result, (y_i) is also basic.

The Paley-Wiener theorem has an immediate corollary which we state here. In this paper, X and Y always denote B-spaces.

COROLLARY 1. *Let $(x_i) \subset X$ be basic and let $(y_i) \subset Y$. Then (y_i) is equivalent to (x_i) if and only if there is an isomorphism $T:\mathrm{sp}(y_i) \to X$ and $\lambda \in (0, 1)$ such that for all finite sequences (a_i) of scalars,*

$$\left\| \sum a_i(x_i - Ty_i) \right\| \leqq \lambda \left\| \sum a_i x_i \right\|.$$

First, we remove the restrictions that T be a linear map and a homeomorphism, but retain equivalence. The results resemble somewhat the approximate solvability theorems of Kantorovich [**4**, p. 543]. The following lemmas are very simple, so the proofs will be omitted.

LEMMA 1. *Let A be a linear subspace of X, $T:A \to Y$ be linear and $\phi: T(A) \to X$. Then T is one-to-one if $\|I - \phi \circ T\| < 1$, where $I = I_A$.*

LEMMA 2. *If $T:A \to Y$ (with A as above) is a bounded linear map, then T is an isomorphism of A into Y if and only if there exists $f:T(A) \to X$, either bounded or continuous, such that $\|I - f \circ T\| < 1$. If there exists $m > 0$ such that $\|f(y)\| \geqq m\|y\|$ for all $y \in T(A)$, boundedness of T is an unnecessary hypothesis.*

This theorem is one generalization of the Paley-Wiener theorem which follows from the lemmas.

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THEOREM 1. *Let* (x_i) *be basic in* X *and* (y_i) *a sequence in* Y. *Then* (y_i) *is equivalent to* (x_i) *if there exist positive constants* m, M *and* λ *with* $\lambda \in (0, 1)$ *and* $(z_i) \subset X$ *such that, for each finite sequence of scalars* (a_i), *there exists a finite sequence of scalars* (b_i) *such that* $\|\sum(a_i x_i - b_i z_i)\| \leqq \lambda \| \sum a_i x_i\|$ *and* $m\| \sum a_i y_i\| \leqq \| \sum b_i z_i\| \leqq M\| \sum a_i y_i\|$.

PROOF. Define $f:\mathrm{sp}(y_i) \to \mathrm{sp}(z_i)$ by $f(\sum a_i y_i) = \sum b_i z_i$ and apply Lemma 2.

In [6], Retherford proves a Paley-Wiener type theorem which does not yield equivalent sequences. It follows as a corollary to the next result.

THEOREM 2. *Let* (x_i) *be basic in* X *and* (y_i) *a sequence in* Y. *Then* (y_i) *is basic if there is a map* $F:\mathrm{sp}(y_i) \to \mathrm{sp}(x_i)$ *and positive constants* m *and* M *such that either* (a) *and* (b) *or* (a') *hold.*

(a) $m\|u\| \leqq \|F(u)\| \leqq M\|u\|$ *for every* $u \in \mathrm{sp}(y_i)$.

(b) If $u = \sum_{i=1}^{p} a_i y_i$, *there exists* (b_i) *such that for every* $q \leqq p$, $F(\sum_{i=1}^{q} a_i y_i) = \sum_{i=1}^{q} b_i x_i$.

(a') *If* $S_n:[x_i] \to [x_i \mid i \leqq n]$ *is the natural projection, for every finite sequence* $(a_i)_{i=1}^{p}$ *and every* $q \leqq p$,

$$m\left\| \sum_{i=1}^{q} a_i y_i \right\| \leqq \left\| S_q\left(F\left(\sum_{i=1}^{p} a_i y_i\right)\right)\right\| \leqq M\left\| \sum_{i=1}^{q} a_i y_i\right\|.$$

PROOF. In either case, the result follows from the inequalities

$$\left\|\left(\sum_{i=1}^{q} a_i y_i\right)\right\| \leqq \frac{1}{m}\left\| S_q\left(F\left(\sum_{i=1}^{p} a_i y_i\right)\right)\right\|$$
$$\leqq \frac{K}{m}\left\| F\left(\sum_{i=1}^{p} a_i y_i\right)\right\| \leqq \frac{KM}{m}\left\|\sum_{i=1}^{p} a_i y_i\right\|$$

where $\|S_n\| \leqq K$, since (y_i) is basic if and only if there exists C such that $\| \sum_{i=1}^{q} a_i y_i\| \leqq C\| \sum_{i=1}^{p} a_i y_i\|$ holds for all $q \leqq p$ and all (a_i).

Retherford's theorem follows from (a').

The interesting point here is that for any pair of basic sequences, (x_i) and (y_i), there is a map F satisfying (a) and (b).

PROPOSITION. *Let* (x_i) *be basic in* X *and let* (e_i) *denote the natural basis of* (l_1). *There is a map* $F:[x_i] \to (l_1)$ *satisfying* (a) *and* (b) *of Theorem* 2.

PROOF. If $(f_k) \subset X^*$ is the sequence biorthogonal to (x_k), let $S_n:[x_k] \to [x_k \mid k \leqq n]$ be the natural projection, i.e.

$$S_n(u) = \sum_{k=1}^{n} f_k(u) x_k.$$

Define a new norm on $[x_i]$ by

$$|x| = (\sup_n \|S_n(x)\|) + \sum 2^{-k} |f_k(x)| \, \|x\|.$$

If we set $A_n(x) = (\operatorname{sgn} f_n(x))(|S_n(x)| - |S_{n-1}(x)|)$, and $F(x) = \sum A_n(x) e_n$, we have $\|F(x)\| = |x|$, as well as (b).

COROLLARY. *If $(x_i) \subset X$ and $(y_i) \subset Y$ are basic sequences, there is a map F:sp$(x_i) \to$sp(y_i) satisfying* (a) *and* (b) *of Theorem* 2.

REMARK 1. In [3], Kadec constructs a homeomorphism between $[x_i]$ and $[y_i]$ with a very delicate refinement of maps similar to F of Theorem 2.

REMARK 2. Pełczynski has recently shown that there is always an isomorphism of $[x_i]$ into a universal space with basis (e_i) such that (x_i) is equivalent to (e_{k_i}) (if $0 < \inf \|x_i\| \leqq \sup \|x_i\| < \infty$), and $[e_{k_i}]$ is complemented in $[e_i]$, [5].

REMARK 3. If (M_i) is a sequence of closed subspaces of X such that each $x \in X$ can be expressed uniquely as $x = \sum u_i$, converging strongly and having $u_i \in M_i$ for each i, then (M_i) is called a *Schauder basis of subspaces* (SBOS) for X. In this paper, most of the results remain valid if the basic sequence (x_i) is replaced by sequences (u_i) with $u_i \in M_i$ for each i and (M_i) a SBOS of X. For example,

PROPOSITION. *Let (M_i) be a SBOS of X and (y_i) a sequence in Y. Then (y_i) is basic if there exists a map F:sp$(y_i) \to X$ and positive constants m and M such that $m\|u\| \leqq \|F(u)\| \leqq M\|u\|$ for all $u \in$sp(y_i), and if for each $u = \sum_{i=1}^{p} a_i y_i$, there exists (z_i) in X with $z_i \in M_i$ for each i such that $q \leqq p$ implies $F(\sum_{i=1}^{q} a_i y_i) = \sum_{i=1}^{q} z_i$.*

The proof is the same as that of Theorem 2 since (M_i) is a SBOS of $[M_i]$ if and only if there exists a constant C with $C \geqq 1$ such that $\|\sum_{i=1}^{q} z_i\| \leqq C \|\sum_{i=1}^{p} z_i\|$ holds if $q \leqq p$ and $z_i \in M_i$ for each i (see, e.g. [6]).

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