ON THE HAUSDORFF OPEN CONTINUOUS IMAGES OF HAUSDORFF PARACOMPACT p-SPACES

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1. **Introduction.** Ponomarev proved the following remarkable theorem: *Every T_0 first-countable space of infinite cardinality is an open continuous image of a zero-dimensional metrizable space of the same weight* [8].[2] This theorem clearly and succinctly summarizes the behavior of metrizable spaces under open mappings. The purpose of this article is to prove an analogue of Ponomarev's theorem in a not necessarily first-countable situation and to develop some of its consequences. This analogue, Theorem 1 below, is a joint discovery of the author and Dr. J. M. Worrell, Jr. [10]. Remark 4 shows how a proof of Ponomarev's theorem may be derived from the proof of Theorem 1. Theorem 1 leads directly to a characterization (Theorem 2) of the class of Hausdorff open continuous images of Hausdorff paracompact p-spaces as the class of Hausdorff spaces of point-countable type. The latter class generalizes the class of Hausdorff first-countable spaces. Both the concept of p-space and of space of point-countable type are due to Arhangel'skiĭ [3], [4]. Theorem 3, a rather direct consequence of Theorem 1, answers a question of Arhangel'skiĭ by generalizing a theorem of his to the Hausdorff case. A relation between Theorem 1, which involves single-valued mappings, and Theorem 3, which involves many-valued mappings, is pointed out in Remark 3.

2. **Terminology.** The general terminology used here is much like that of [7], one exception being that spaces called *compact* in [7] are here called *bicompact*. The usage of [7] in letting X ambiguously denote the topological space $(X, \mathfrak{J})$ is followed where convenient, and *product space* refers to a Cartesian product of spaces endowed with the product topology [7]. A *base for X* means a base for the topology of X. The letter N denotes the set of positive integers and if A is a set, $\aleph(A)$ denotes the cardinal number of A. The *weight* [2] of a topological space $(X, \mathfrak{J})$ is defined as the smallest cardinal num-

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[2] Ponomarev does not point out that infinite cardinality is required. In fact, if S is a finite T_0 but not T_1 space any T_1 open continuous preimage of S has infinite weight and cardinality. In Theorem 1 infinite cardinality is not required since the spaces here are assumed to be T_2.

ber m such that $\mathfrak{I}$ has a base of cardinal m. A mapping $f: X \to Y$ is called *perfect* [1] if and only if it is closed, continuous, and $f^{-1}(y)$ is bicompact for all $y \in Y$. If $\mathfrak{A}$ is a collection of sets then $\mathrm{St}(x, \mathfrak{A})$ denotes $\bigcup\{A \in \mathfrak{A}: x \in A\}$. A T_1-space X is called a *p-space* [3] if and only if there exists a sequence $\mathcal{G}_1, \mathcal{G}_2, \cdots$ of collections of open subsets of the Wallman bicompactification ωX of X covering X such that if $x \in X$, $\bigcap\{\mathrm{St}(x, \mathcal{G}_n): n \in N\} \subset X$. If X is a Tychonoff space this definition is equivalent to one in which βX (the Stone-Čech bicompactification of X) replaces ωX. A principal theorem for p-spaces, suggestive of the naturality of their use in Theorem 1, is that of Arhangel'skiĭ: *A T_2-space is a paracompact p-space if and only if there exists a perfect mapping of it onto a metrizable space* [3, Theorem 5.1].

3. **Theorems.** If $(X, \mathfrak{I})$ is a space and $A \subset X$, a subcollection $\mathfrak{D}$ of $\mathfrak{I}$ whose members include A is called a *base at A* if and only if for every $U \in \mathfrak{I}$ such that $U \supset A$, there exists $D \in \mathfrak{D}$ such that $A \subset D \subset U$.

If X is a space and $A \subset X$, then A is said to be of *countable character* [4] if and only if there exists a countable base at A.

A space X is said to be of *point-countable type* [4] if and only if X is covered by a collection of bicompact subspaces of countable character.

REMARK 1. Any first-countable space is of point-countable type.

REMARK 2. The property of being of point-countable type is preserved by open continuous mappings.

The following lemma was stated by Arhangel'skiĭ [5, p. 158]. A proof is sketched here for completeness.

LEMMA 1. *A Tychonoff p-space is of point-countable type.*

PROOF. Every point of such a space X lies in a bicompact subset of X which is a G_δ-set in βX and every such set has countable character.

LEMMA 2. *In a Hausdorff space X the following properties are equivalent:*

(i) *X is of point-countable type.*

(ii) *If U is open in X and $x \in U$ there exists a bicompact set B of countable character such that $x \in B \subset U$.*

PROOF. Clearly (ii) implies (i). Suppose $x \in U$ and U is open. There exists a bicompact set B of countable character containing x. Let $\{U_k: k \in N\}$ be a base at B such that $U_{k+1} \subset U_k$ for all $k \in N$. Then since X is Hausdorff $B = \bigcap\{\overline{U}_k: k \in N\}$. Let $V_1 = U$. Suppose open sets $V_1, \cdots, V_n$ have been defined such that $x \in V_k \subset U_k \cap V_{k-1}$ and $\overline{V}_k$ is disjoint from $B \sim V_{k-1}$ for $1 < k \leqq n$. Since $B \sim V_n$ is bicompact, $x \in V_n$ and X is T_2, there exists an open set V such that $x \in V \subset \overline{V} \subset X \sim (B \sim V_n)$. Let $V_{n+1} = V \cap V_n \cap U_{n+1}$. Thus there exists a sequence

$\{V_n\}$ such that for all $n\in N$, $x\in V_{n+1}\subset V_n\cap U_{n+1}$ and $\overline{V}_{n+1}$ is disjoint from $B\sim V_n$. Let $C=\bigcap\{\overline{V}_n: n\in N\}$. Then C is a closed (therefore bicompact) subset of B containing x. Since $\overline{V}_{n+1}\subset(X\sim B)\cup V_n$, $C=\bigcap\{V_n: n\in N\}$ and $C\subset U$. Suppose W is open and $C\subset W$. If no $V_n\subset W$, there exists a sequence $\{x_k\}$ such that each $x_k\in V_k\sim W$. Since $\bigcap\{\overline{V}_k\sim W: k\in N\}=\emptyset$ and B is bicompact, there exists n such that $\overline{V}_k\sim W\subset X\sim B$ for all $k\geqq n$. Let $A=\{x_k: k\geqq n\}$. Then $\overline{A}\subset X\sim W$ and $\overline{A}\cap B\neq\emptyset$. For if $B\subset X\sim\overline{A}$, then for some $k\geqq n$, $U_k\subset X\sim\overline{A}\subset X\sim A$ contradicting $x_k\in A$. If $y\in\overline{A}\cap B$, $y\in\overline{V}_k\sim W$ for all $k\in N$, again a contradiction. Hence some $V_n\subset W$, so that C has countable character.

THEOREM 1. *Suppose X is a Hausdorff space of point-countable type. Then X is the range of an open continuous mapping ϕ such that:* (1) *The domain Y of ϕ is a Hausdorff paracompact p-space.* (2) *The weight of Y is the weight of X.* (3) *Y is a subspace of the product space of a zero-dimensional metrizable space and X.*

PROOF. See §4.

COMMENT. For Tychonoff spaces, part (1) can be derived from [4, Theorem 3.14] by the method of Remark 3 below.

THEOREM 2. *A Hausdorff space is of point-countable type if and only if it is an open continuous image of a Hausdorff paracompact p-space.*

PROOF. This follows from Theorem 1, Lemma 1, and Remark 2.

Recall that a many-valued mapping $f: X\rightarrow Y$ is called *continuous* (*from above*) [9] if and only if for every $x\in X$ if $V\subset Y$ is open and $fx\subset V$ there exists an open $U\subset X$ such that $x\in U$ and $f(U)\subset V$. The mapping f is called *range-bicompact* (or *Y-bicompact* [9]) if and only if fx is bicompact for every $x\in X$. Arhangel'skiĭ proved the following theorem with the additional hypothesis that X is a Tychonoff space [4, Theorem 3.14] and asked [4, p. 54] whether it is valid for a wider class of spaces.

THEOREM 3. *Suppose X is a Hausdorff space. Then X is of point-countable type if and only if X is the range of an open continuous* (*possibly many-valued*) *range-bicompact mapping of a metrizable space.*

PROOF. By Theorem 1, there exists a continuous mapping ϕ of a T_2 paracompact p-space Y onto X. By Arhangel'skiĭ's theorem (see §2) there exists a perfect mapping θ of Y onto a metrizable space M. It is straightforward to show that $\phi\circ\theta^{-1}$ is an open continuous range-bicompact mapping of M onto X. The sufficiency follows from [4, Proposition 3.6].

REMARK 3. Theorem 3 can be used to derive part (1) of Theorem 1. For if f is an open continuous many-valued range-bicompact mapping of a metrizable space X onto a Hausdorff space Y of point-countable type, let $Z=\{(x, y)\in X\times Y: y\in fx\}$, under the topology induced by the product topology. The set Z is called the *graph* of f by Ponomarev [9]. If θ and ϕ denote the projections of Z onto X and Y respectively, then it may be seen that $f=\phi\circ\theta^{-1}$ where ϕ is open and continuous and θ is perfect. (This statement may be proved in a fashion similar to that used by Ponomarev in showing that a perfect mapping f factors into $\phi\circ\theta^{-1}$ where θ and ϕ are perfect [9, Theorem 1, §2].) Hence Z is a paracompact p-space by Arhangel'skiĭ's theorem and ϕ maps Z onto Y.

4. Proof of Theorem 1.

PROOF. Assume $\aleph(X)$ is infinite. Let $\mathcal{C}$ denote $\{B\subset X: B$ is bicompact and of countable character$\}$. For some base $\mathcal{W}$ of X such that weight of $X=\aleph(\mathcal{W})$, let $\mathcal{F}$ denote the collection of all unions of finite subcollections of $\mathcal{W}$. Then $\aleph(\mathcal{F})=$ weight of X and $\mathcal{W}\subset\mathcal{F}$. Call a sequence α *admissible* if and only if for each $n\in N$: (1) $\alpha(n)\in\mathcal{F}$; (2) $\alpha(n+1)\subset\alpha(n)$; (3) for some $B\in\mathcal{C}$, $B=\bigcap\{\alpha(k): k\in N\}$ and $\{\alpha(k): k\in N\}$ is a base at B. Using bicompactness it may be seen that for each $B\in\mathcal{C}$ there exists an admissible sequence α satisfying (3) with respect to B.

Consider $\mathcal{F}$ as a topological space with the discrete topology and let Δ denote the product space of countably many copies of $\mathcal{F}$. Let $\Gamma=\{\alpha\in\Delta: \alpha$ is admissible$\}$. Then Γ is a metrizable zero-dimensional space (it is a subspace of a Baire space [6]). Let $\Gamma\times X$ denote the product space of Γ and X and let

$$Y = \{(\alpha, x) \in \Gamma \times X : x \in \bigcap\{\alpha(k): k \in N\}\},$$

with the topology induced by the product topology. Note that Y is Hausdorff. Let $\theta=\pi_1|Y$ and $\phi=\pi_2|Y$, where π_i denotes projection onto the ith coordinate. Then θ and ϕ are continuous mappings of Y onto Γ and X respectively.

If $\alpha\in\Gamma$, let $S(\alpha|n)=\{\alpha'\in\Gamma: \alpha'(k)=\alpha(k),\ k=1, \cdots, n\}$. Then $\{S(\alpha|n): n\in N$ and $\alpha\in\Gamma\}$ is a base for Γ. For $\alpha\in\Gamma$ and $V\in\mathcal{F}$ such that $V\subset\alpha(n)$ let $D(\alpha|n;\ V)$ denote $(S(\alpha|n)\times V)\cap Y$. Then $\mathcal{B}=\{D(\alpha|n;\ V): \alpha\in\Gamma,\ n\in N,\ V\in\mathcal{F}$, and $V\subset\alpha(n)\}$ is a base for Y. Since $\aleph(\mathcal{F})=$ weight of X, $\aleph(\mathcal{B})=$ weight of X.

Suppose $\alpha\in\Gamma$, $V\in\mathcal{F}$, and $V\subset\alpha(n)$. Then clearly $\phi[D(\alpha|n;\ V)]\subset V$. If $x\in V$, then by Lemma 2 there exists $B\in\mathcal{C}$ such that $x\in B\subset V$. Let $\beta\in\Gamma$ be such that $\{\beta(k): k\in N\}$ is a base at B. There exists k such that $\beta(k)\subset V$. The sequence α' such that $\alpha'(j)=\alpha(j)$, $1\leq j\leq n$

and $\alpha'(j)=\beta(k+j)$ for $j>n$ is admissible and $(\alpha', x)\in D(\alpha|n; V)$. Hence $\phi[D(\alpha|n; V)]=V$. Therefore ϕ is an open mapping.

If it is shown that θ is a perfect mapping, then by Arhangel'skiĭ's theorem cited in §2, Y is a paracompact p-space. Suppose $\alpha\in\Gamma$ and $B=\bigcap\{\alpha(k): k\in N\}$. Then, since B is bicompact, $\theta^{-1}(\alpha)=\{\alpha\}\times B$ is bicompact. Hence θ is a bicompact mapping. To show that θ is closed suppose W is open in Y and $\theta^{-1}(\alpha)\subset W$. There exist $m\in N$ and sets $D_k=D(\alpha_k|n(k); V_k)\in\mathfrak{B}$ intersecting $\theta^{-1}(\alpha)$ for $k=1, \cdots, m$, such that $\theta^{-1}(\alpha)\subset\bigcup\{D_k: k\leqq m\}\subset W$. Since $\theta^{-1}(\alpha)$ meets each D_k, $\alpha_k(j)=\alpha(j)$, $1\leqq j\leqq n(k)$, $1\leqq k\leqq m$. Also $B\subset\bigcup\{V_k: k\leqq m\}$. By conditions (2) and (3) on admissible sequences there exists $n\geqq\max\{n(k): k\leqq m\}$ such that $B\subset\alpha(n)\subset\bigcup\{V_k; k\leqq m\}$. If $(\alpha', x)\in D=D(\alpha|n; \alpha(n))$, then $x\in V_k$ for some k, and therefore $(\alpha', x)\in D_k\subset W$. Hence $\theta^{-1}(\alpha)\subset D\subset W$. Since any $\theta^{-1}(\alpha')$ intersecting D is a subset of D, $D=\theta^{-1}\theta(D)$. It follows that θ is a closed mapping.

REMARK 4. If the space X is T_0 and first-countable, then $\mathfrak{C}$ in the above proof can be taken as the collection $\{\{x\}: x\in X\}$. Then each admissible sequence α is such that $\bigcap\{\alpha(k): k\in N\}=\{x\}$ for some $x\in X$. It follows that Y is homeomorphic to Γ and thus X is an open continuous image of Γ. This proves Ponomarev's theorem.

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