A NOTE ON WALLMAN SPACES

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In [3] Orrin Frink introduced the notion of a Wallman compactification of a Tychonoff space. A. K. and E. F. Steiner [6] have recently shown that any compact metric space (or product of compact metric spaces) is a Wallman compactification of each of its dense subspaces. Their result is an immediate consequence of earlier work and a theorem they proved about the existence of a certain kind of base of closed sets in a compact metric space. The purpose of this note is to give a different proof of this latter result, obtaining as a consequence a slightly stronger version.

Definition. Let X be a Tychonoff space. A family $\mathcal{Z}$ of closed subsets of X is said to be a regular normal base for X if $\mathcal{Z}$ is a base for the closed subsets of X, if each member of $\mathcal{Z}$ is a regular set (the closure of its interior) and if

(i) $\mathcal{Z}$ is closed under finite unions and intersections; (i.e., $\mathcal{Z}$ is a ring)

(ii) if $x \in X$, F a closed subset of X, $x \notin F$, then there exists $Z \in \mathcal{Z}$ such that $x \in Z \subset X \backslash F$;

and

(iii) if $A, B \in \mathcal{Z}$, $A \cap B = \emptyset$ then there exists $C, D \in \mathcal{Z}$ such that

$$A \subset X\backslash C, \quad B \subset X\backslash D \quad \text{and} \quad (X\backslash C) \cap (X\backslash D) = \emptyset.$$

If X is a compact metric space, then (ii) and (iii) will hold for $\mathcal{Z}$ if $\mathcal{Z}$ is a base for closed sets and is a ring.

Our result is the following

Theorem. *Let X be a compact metric space and let $\mathcal{R}$ be any countable ring of regular closed subsets of X. Then there is a regular normal base $\mathcal{Z}$ for X such that $\mathcal{Z} \supseteq \mathcal{R}$.*

We notice that the result may fail if $\mathcal{R}$ is not countable. For instance, let $X = [0, 1]$, $\mathcal{R} = \{[0, \alpha] : 0 < \alpha \leq 1\}$. Then any closed set not containing 0 must intersect some member of $\mathcal{R}$ in a single point and $\mathcal{R}$ can not be extended to a normal base for X.

Proof of the Theorem. From now on, let X be a compact metric space. Let $I = [0, 1]$ and let $P = \prod_{i=1}^{\infty} I$. We may assume that $X \subset P$. If $p \in P$ we will write $p = (p_1, p_2, \cdots)$.

Let Z^+ denote the set of positive integers. Let $\mathcal{F}$ denote the set of all functions whose domains are subsets of Z^+ and whose ranges are

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contained in I. (For convenience, we will also include the "function" ψ whose domain is empty.) If α is such a function then $|\alpha|$ will denote the domain of α and $\#|\alpha|$ the number of elements in $|\alpha|$ if it is finite. If α and β are such that $|\alpha|\cap|\beta|=\varnothing$, then by $\alpha\vee\beta$ we will mean the function whose domain is $|\alpha|\cup|\beta|$ and which is defined in the obvious manner. We will also identify points in P with functions in $\mathfrak{F}$ whose domains are Z^+. If $\alpha\in\mathfrak{F}$ and $\#|\alpha|=n$, we will write $|\alpha| =\{\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_n\}$ where $\bar{\alpha}_1<\bar{\alpha}_2<\cdots<\bar{\alpha}_n$.

Let $\mathcal{S}$ denote the set of all finite sequences of ± 1. If $s\in\mathcal{S}$, then $\#s$ will denote the length of s. Let $\alpha\in\mathfrak{F}$ with $\#|\alpha|=n$ and $s\in\mathcal{S}$ with $\#s=n$. The quadrant $Q(\alpha, s)$ determined by α and s is defined by (where $s=(s_1, \cdots, s_n)$)

$$Q(\alpha, s) = \{y\in P: s_j y_{\bar{\alpha}_j} > s_j\alpha(\bar{\alpha}_j) \quad \text{for } 1\leqq j\leqq n\}.$$

If σ is the sequence of length 0, then by the above definition, or by convention, $Q(\psi, \sigma)=P$.

Let $\alpha\in\mathfrak{F}$ with $|\alpha|$ finite. Then α is *admissible* for a closed subset B of P if for every $\beta\in\mathfrak{F}$, $|\alpha|\cap|\beta|=\varnothing$, $|\alpha|\cup|\beta|=Z^+$ such that $\alpha\vee\beta\in B$ and for every neighborhood N (relative to P) of $\alpha\vee\beta$ and every $s\in\mathcal{S}$ with $\#s=\#|\alpha|$ we have $B\cap N\cap Q(\alpha, s)\neq\varnothing$. Notice that ψ is admissible.

The basic result is the following:

LEMMA. *Let $\alpha\in\mathfrak{F}$ be admissible for B. Let $k\notin|\alpha|$ be a fixed positive integer. Then the set of $\beta\in\mathfrak{F}$ with $|\beta|=\{k\}$ such that $\alpha\vee\beta$ is not admissible for B is countable. In particular, the set of α with $\#|\alpha|=1$ which are not admissible for B is countable.*

We will now show that the lemma implies our theorem. Let $\mathfrak{R}$ be a countable regular ring. By induction and the use of our lemma applied to X and the members of $\mathfrak{R}$ we may prove the existence of countable sets $D_1, D_2, \cdots, D_n, \cdots$ such that

(i) for each j, $D_j\subset\mathfrak{F}$ and if $\alpha\in D_j$ then $|\alpha|=\{j\}$;

(ii) $\{\alpha(j):\alpha\in D_j\}$ is dense in I for each j;

(iii) if $\alpha\in D_{n_1}\vee\cdots\vee D_{n_k}$ ($n_i\neq n_j$ if $i\neq j$) then α is admissible for X and for each $B\in\mathfrak{R}$.

For each j we may write $D_j=D_j'\cup D_j''$ where $D_j'\cap D_j''=\varnothing$ and both $\{\alpha(j):\alpha\in D_j'\}$ and $\{\beta(j):\beta\in D_j''\}$ are dense in I. Let $\mathfrak{R}_j =\{[\alpha(j), \beta(j)]:\alpha\in D_j', \beta\in D_j''\}$. Let

$$\mathfrak{R}_1 = \left\{\prod_{j=1}^{\infty} E_j: E_j = (-\infty, \infty) \text{ for all but finitely many } j \text{ and } E_j = \mathfrak{R}_j \text{ for the remaining } j\right\}.$$

Let $\tilde{\mathfrak{R}}_2$ consist of all finite unions of members of $\tilde{\mathfrak{R}}_1$ and let $\mathcal{Z}=$ the ring generated by $\mathfrak{R}\cup\{E\cap X: E\in\tilde{\mathfrak{R}}_{22}\}$. That $\mathcal{Z}$ is a normal base for X follows easily from the density of $\{\alpha(j):\alpha\in D_j'\}$ and $\{\beta(j):\beta\in D_j''\}$ for all j. That the members of $\mathcal{Z}$ are regular follows from the admissibility of all $\alpha\in D_{n_1}\vee\cdots\vee D_{n_k}$ for X and for all $B\in\mathfrak{R}$ and the fact that if A and B are regular and $A\cap B$ is regular in B then $A\cap B$ is regular.

We now turn to the proof of the lemma. Let α be admissible and let $k\notin|\alpha|$ be a positive integer. Suppose $\#|\alpha|=n$. Let $e>0$, $s\in\mathcal{S}$ with $\#s=n+1$ and an integer $m\geqq\max(\bar{\alpha}_n, k)$ be fixed. For simplicity of notation, assume that the sequence s is indexed with the elements of $|\alpha|\cup\{k\}$ in increasing order. Let $\bar{s}$ be the sequence s with the term s_k (under this new labelling) deleted, and assume that $s_k=+1$. For $u\in P$ let

$$C_u^e=\{y\in P: |u_i-y_i|<e,\ i=1, 2, \cdots, m\}.$$

Suppose there exists $\beta\in\mathfrak{F}$ with $|\beta|=\{k\}$ such that

(i) there exists $\gamma\in\mathfrak{F}$ with $|\gamma|\cap(|\alpha|\cup\{k\})=\varnothing$ and $|\alpha|\cup\{k\}\cup|\gamma|=Z^+$, such that $p=\alpha\vee\beta\vee\gamma\in B$, and

(ii) $B\cap C_p^e\cap Q(\alpha\vee\beta, s)=\varnothing$.

We claim that the set of such β is finite. This will show that the number of β such that $\alpha\vee\beta$ is not admissible is countable. Suppose $\beta'\neq\beta$ is such that there exists γ' as in (i) such that $p'=\alpha\vee\beta'\vee\gamma'$ satisfies (i) and (ii) above. We claim that $C_p^{e/4}\cap C_{p'}^{e/4}=\varnothing$. This will clearly prove the result.

Suppose that $q\in C_p^{e/4}\cap C_{p'}^{e/4}$. Assume that $\beta(k)>\beta'(k)$. Then $|\beta(k)-q_k|<e/4$ and $|\beta'(k)-q_k|<e/4$ so that $|\beta(k)-\beta'(k)|<e/2$. Let $\delta=\beta(k)-\beta'(k)$. Then $0<\delta<e/2$. Since α is admissible, there exists $r\in B\cap Q(\alpha, \bar{s})\cap C_p^\delta$. We claim that $r\in B\cap C_{p'}^e\cap Q(\alpha\vee\beta', s)$, in contradiction of (ii) above, thus proving the result. Certainly $r\in B$. Furthermore, since $r\in Q(\alpha, \bar{s})$ and $|r_k-\beta(k)|<\delta=|\beta(k)-\beta'(k)|$ we certainly have that $r_k>\beta'(k)$ so that $r\in Q(\alpha\vee\beta', s)$. Finally, to show that $r\in C_{p'}^e$ we have to show that $|r_j-p_j'|<e$ for $j=1, \cdots, m$. But $|r_j-p_j|<\delta$ and

$$|p_j-p_j'|\leqq|p_j-q_j+q_j-p_j'|<e/4+e/4=e/2,$$

so the result follows by the triangle inequality.

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