LEBESGUE CHARACTERIZATIONS OF UNIFORMITY-DIMENSION FUNCTIONS

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1. **Introduction.** Let (X, p) be a metric space, let $\dim(X)$ be the covering dimension of X, and let $d_0(X, p)$ be the metric dimension of X. Let d_2 and d_3 denote the metric-dependent dimension functions introduced by Nagami and Roberts [7], and let d_6 and d_7 be the metric-dependent dimension functions introduced by Smith [9]. Characterizations of some of these metric-dependent dimension functions in terms of Lebesgue covers have been given by Egorov [1], Wilkinson [11] and Smith [9]. These results are described by the following table.

Metric-dependent dimension function	Characterization
$d_2(X, p) \leqq n$	Every Lebesgue cover consisting of $n+2$ members has an open refinement of order $\leqq n+1$
$d_3(X, p) \leqq n$	Every finite Lebesgue cover has an open refinement of order $\leqq n+1$
$d_6(X, p) \leqq n$	Every countable Lebesgue cover has an open refinement of order $\leqq n+1$
$d_7(X, p) \leqq n$	Every locally finite Lebesgue cover has an open refinement of order $\leqq n+1$
$d_0(X, p) \leqq n$	Every Lebesgue cover has an open refinement of order $\leqq n+1$

Soniat [10] has generalized the dimension functions d_0, d_2 and d_3 for uniform spaces and obtained Lebesgue-type characterizations for d_3 and d_0. In this paper we complete the above characterization table for uniform spaces. In §2 we develop Lebesgue cover properties for uniform spaces and characterize d_2. In §§3 and 4 we generalize the dimension functions d_6 and d_7 to uniform spaces and characterize them in terms of Lebesgue covers.

Definition. Let X be a set and $\mathfrak{D} = \{\mathfrak{D}_\lambda : \lambda \in A\}$ be a family of collections of subsets of X. For each $\lambda \in A$, let $\mathfrak{D}_\lambda = \{D_\alpha : \alpha \in A_\lambda\}$. Then

$$\bigwedge_{\lambda \in A} \{\mathfrak{D}_\lambda\} = \{\bigcap D_{\alpha(\lambda)} : \alpha(\lambda) \in A_\lambda, \lambda \in A\}.$$

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DEFINITION. Throughout this paper J will denote the set $\{1, 2, \cdots, n+1\}$ and $J' = J \cup \{n+2\}$, where the integer n will always be understood.

2. Characterization of d_2 for uniform spaces.

The reader is referred to the papers by Nagami and Roberts [7], Smith [9], and Soniat [10] for the definitions of the dimension functions d_0, d_2, d_3, d_6 and d_7 and the generalizations of d_0, d_2, and d_3 to uniform spaces.

DEFINITION 2.1. Let C and C′ be subsets of a uniform space $(X, \mathfrak{U})$. We say that C and C′ are *separated* provided there exists $U \in \mathfrak{U}$ such that $(C \times C') \cap U = \varnothing$. If $\mathfrak{C} = \{C_\alpha, C'_\alpha : \alpha \in A\}$ is a family of pairs (C_α, C'_α), then $\mathfrak{C}$ is called *uniformly separated* if there exists $U \in \mathfrak{U}$ such that $(C_\alpha \times C'_\alpha) \cap U = \varnothing$ for all $\alpha \in A$.

DEFINITION 2.2. A cover $\mathfrak{D}$ of a uniform space $(X, \mathfrak{U})$ is called *Lebesgue* if there exists $U \in \mathfrak{U}$ such that $\{U(x) : x \in X\}$ refines $\mathfrak{D}$.

DEFINITION 2.3. A cover $\mathfrak{D} = \{D_\alpha : \alpha \in A\}$ of a uniform space $(X, \mathfrak{U})$ is called *$\mathfrak{U}$-shrinkable* if there exists some $U \in \mathfrak{U}$ and a cover $\mathfrak{F} = \{F_\alpha : \alpha \in A\}$ such that

(1) $F_\alpha \subset D_\alpha$ for all $\alpha \in A$.

(2) $\{F_\alpha, X - D_\alpha : \alpha \in A\}$ *is uniformly separated by U.*

THEOREM 2.4. *A cover $\mathfrak{D}$ of a uniform space $(X, \mathfrak{U})$ is Lebesgue if and only if $\mathfrak{D}$ is $\mathfrak{U}$-shrinkable.*

PROOF (Necessity). Let $\mathfrak{D} = \{D_\alpha : \alpha \in A\}$ be a Lebesgue cover of $(X, \mathfrak{U})$. Then there exists $U \in \mathfrak{U}$ such that $\{U(x) : x \in X\}$ refines $\mathfrak{D}$. Choose $V \in \mathfrak{U}$ such that V is symmetric and $V^2 \subset U$. Define $F_\alpha = \{x : V(x) \cap (X - D_\alpha) = \varnothing\}$ for $\alpha \in A$.

(i) We assert that $\{F_\alpha : \alpha \in A\}$ covers X. Clearly $F_\alpha \subset D_\alpha$ for all $\alpha \in A$. Let $x \in D_\alpha - F_\alpha$. Then since $\{U(x) : x \in X\}$ refines $\mathfrak{D}$, there exists $\beta \in A$ such that $V(x) \subset U(x) \subset D_\beta$. Hence $V(x) \cap (X - D_\beta) = \varnothing$ so that $x \in F_\beta$.

(ii) We now assert that $\{F_\alpha, X - D_\alpha : \alpha \in A\}$ is uniformly separated by V. Suppose there exists some $\beta \in A$ such that $[F_\beta \times (X - D_\beta)] \cap V \neq \varnothing$. Let $(x, y) \in [F_\beta \times (X - D_\beta)] \cap V$. Then $x \in F_\beta$, $y \in X - D_\beta$ and $X \in V(y)$ and $y \in V(x)$. But $x \in F_\beta$ implies $V(x) \cap [X - D_\beta] = \varnothing$ so $y \notin V(x)$, a contradiction.

REMARK. It should be noted at this point that the cover $\mathfrak{F} = \{F_\alpha : \alpha \in A\}$ defined above is Lebesgue. If $x \in X$, then there exists $\beta \in A$ such that $x \in U(x) \subset D_\beta$. Clearly $x \in F_\beta$ and we assert that $V(x) \subset F_\beta$. Let $y \in V(x)$ and $z \in V(y)$, so that $(x, y) \in V$ and $(y, z) \in V$. Hence $(x, z) \in V^2 \subset U$ and therefore $z \in U(x) \subset D_\beta$. It now follows that $V(y) \cap (X - D_\beta) = \varnothing$ and $y \in F_\beta$. Thus $V(x) \subset F_\beta$.

(Sufficiency). Suppose $\mathfrak{D}=\{D_\gamma:\alpha\in A\}$ is $\mathfrak{U}$-shrinkable to $\mathfrak{F}=\{F_\alpha:\alpha\in A\}$, where $\{F_\alpha, X-D_\alpha:\alpha\in A\}$ is uniformly separated by symmetric $U\in\mathfrak{U}$. Let $x\in X$. Since $\mathfrak{F}$ is a cover of X, there exists $\beta\in A$ such that $x\in F_\beta$. Let $y\in U(x)$ so that $(x, y)\in U$. But

$$U\cap[F_\beta\times(X-D_\beta)]=\varnothing$$

and hence $y\notin X-D_\beta$. Thus $y\in D_\beta$ and $U(x)\subset D_\beta$. Therefore $\{U(x):x\in X\}$ refines $\mathfrak{D}$ and $\mathfrak{D}$ is Lebesgue.

For normal uniform spaces $(X, \mathfrak{U})$ Soniat has shown the following [**10**, Theorem 3.8].

THEOREM 2.5. *Let $(X, \mathfrak{U})$ be a normal uniform space. Then $d_2(X, \mathfrak{U})\leqq n$ if and only if for every uniformly separated collection $\{C_i, C_i':i\in J\}$ of closed sets, there exists a closed collection $\{B_i:i\in J\}$ such that B_i separates C_i and C_i' and $\bigcap_{i\in J}B_i=\varnothing$.*

Note. Here the separating sets B_i are subsets of X and separate C_i and C_i' in the usual sense and are not to be confused with elements of the uniformity $\mathfrak{U}$.

The next theorem now follows directly from [**9**, Theorem 2.3] where the Lebesgue covers are now in the uniformity sense rather than the metric sense.

THEOREM 2.6. *Let $(X, \mathfrak{U})$ be a completely normal uniform space. Then $d_2(X, \mathfrak{U})\leqq n$ if and only if for every collection $\{\mathfrak{D}_i:i\in J\}$ of $n+1$ binary Lebesgue covers of X, the cover $\mathfrak{D}=\bigwedge_{i\in J}\mathfrak{D}_i$ of X has an open refinement of order $\leqq n+1$.*

We now obtain a Lebesque characterization of $(X, \mathfrak{U})$ analogous to [**9**, Theorem 2.4].

THEOREM 2.7. *Let $(X, \mathfrak{U})$ be a completely normal uniform space. Then $d_2(X, \mathfrak{U})\leqq n$ if and only if every Lebesgue cover $\mathfrak{D}=\{D_1, D_2, \cdots, D_{n+2}\}$ of X consisting of $n+2$ members has an open refinement of order $\leqq n+1$.*

PROOF (Necessity). Suppose $d_2(X, \mathfrak{U})\leqq n$, and let $\mathfrak{D}=\{D_1, D_2, \cdots, D_{n+2}\}$ be a Lebesgue cover of X. Then there exists $U\in\mathfrak{U}$ such that $\{U(x):x\in X\}$ refines $\mathfrak{D}$. By Theorem 2.4 above we can uniformly shrink $\mathfrak{D}$ to a closed Lebesgue cover $\mathfrak{F}=\{F_1, F_2, \cdots, F_{n+2}\}$ such that $F_i\subset D_i$ for $i\in J'$. Then for each $i\in J$, $\mathfrak{D}_i=\{D_i, X-F_i\}$ is a binary Lebesgue cover of X. By Theorem 2.6 above $\mathfrak{D}^*=\bigwedge_{i\in J}\mathfrak{D}_i$ has an open refinement $\mathfrak{D}^{**}$ such that $\mathrm{ord}(\mathfrak{D}^{**})\leqq n+1$. But $\mathfrak{D}^*$ refines $\mathfrak{D}$ since $\mathfrak{F}$ covers X. Hence $\mathfrak{D}^{**}$ is the desired open cover.

(Sufficiency). Let $\{C_i, C_i' : i \in J\}$ be a collection of $n+1$ pairs of closed sets which are uniformly separated by $U \in \mathfrak{U}$. Choose K and V symmetric in $\mathfrak{U}$ such that $K \subset K^2 \subset V \subset V^2 \subset U$. Now define $\mathfrak{K} = \{K(x) : x \in X\}$ and $\mathfrak{V} = \{V(x) : x \in X\}$. Define for each $i \in J$, $D_i = \mathrm{St}(C_i, \mathfrak{V})$ and $H_i = [\mathrm{St}(C_i, \mathfrak{K})]^-$ where $\mathrm{St}(C_i, \mathfrak{V})$ is the star of C_i with respect to the cover $\mathfrak{V}$. Let $D_{n+2} = X - \bigcup_{i \in J} H_i$. Clearly $\mathfrak{D} = \{D_1, D_2, \cdots, D_{n+2}\}$ is an open Lebesgue cover of X. Hence $\mathfrak{D}$ has an open refinement $\mathfrak{R} = \{R_\alpha : \alpha \in A\}$ such that the $\mathrm{ord}(\mathfrak{R}) \leqq n+1$. Define f to be the function, $f: A \to J'$, such that

$$f(\alpha) = \{\text{smallest integer } i \in J' \text{ such that } R_\alpha \subset D_i\}.$$

Now define $R_i = \bigcup \{R_\alpha : f(\alpha) = i\}$ for each $i \in J'$. Hence $\mathfrak{R} = \{R_1, R_2, \cdots, R_{n+2}\}$ may replace $\{R_\alpha : \alpha \in A\}$. Choose $K^* \in \mathfrak{U}$ such that $(K^*)^2 \subset K$ and define

$$\mathfrak{K}^* = \{K^*(x) : x \in X\}, \qquad E_i = \{x : x \in C_i, x \notin R_i\},$$

$$S_i = \mathrm{St}(E_i, \mathfrak{K}^*), \qquad R_i^* = R_i \cup S_i, \quad \text{for } i \in J, \quad \text{and} \quad R_{n+2}^* = R_{n+2}.$$

Now $S_i \cap D_{n+2} = \varnothing$; for $x \in S_i$ implies that $x \in \mathrm{St}(E_i, \mathfrak{K}^*) \subset \mathrm{St}(C_i, \mathfrak{K}) \subset H_i$ so that $x \notin D_{n+2}$. Hence $\mathfrak{R}^* = \{R_1^*, R_2^*, \cdots, R_{n+2}^*\}$ is an open cover of X such that ord $(\mathfrak{R}^*) \leqq n+1$ and $C_i \subset R^*$ for $i \in J$. Since $\mathfrak{R}^*$ is finite there exists by [**5**, Lemma 1.5] a closed cover $\mathfrak{F} = \{F_1, F_2, \cdots, F_{n+2}\}$ of X such that $C_i \subset F_i \subset R_i^*$ for $i \in J$, and $F_{n+2} \subset R_{n+2}^*$. X normal implies that there exist open sets 0_i such that $F_i \subset 0_i \subset \bar{0}_j \subset R_j^*$ for $i \in J$. Define $B_i = \bar{0}_i - 0_i$ for $i \in J$. Clearly B_i is a closed set separating C_i and C_i' for $i \in J$. We assert $\bigcap_{i \in J} B_i = \varnothing$. Suppose there exists $x \in \bigcap_{i \in J} B_i$. Then $x \notin F_i$ for each $i \in J$. Hence $x \in F_{n+2} \subset R_{n+2}^*$. But $x \in R_i^*$ for all $i \in J$ and hence $x \in \bigcap_{i \in J'} R_i^*$. This is a contradiction since $\mathrm{ord}(\mathfrak{R}^*) \leqq n+1$. Hence $d_2(X, \mathfrak{U}) \leqq n$.

3. The uniformity dimension function d_6

DEFINITION 3.1. Let $(X, \mathfrak{U})$ be a uniform space. If $X = \varnothing$, $d_6(X, \mathfrak{U}) = -1$. Otherwise $d_6(X, \mathfrak{U}) \leqq n$ if $(X, \mathfrak{U})$ satisfies this condition:

(D_6) Given any countable collection of closed pairs $\{C_i, C_i' : i = 1, 2, \cdots\}$ such that

(1) $\{C_i, C_i' : i = 1, 2, \cdots\}$ is uniformly separated,

(2) $\{X - C_i' : i = 1, 2, \cdots\}$ is locally finite,

then there exist closed sets B_i separating C_i and C_i' such that ord $\{B_i : i = 1, 2, \cdots\} \leqq n$.

THEOREM 3.2. *Let $(X, \mathfrak{U})$ be a paracompact uniform space. Then $d_6(X, \mathfrak{U}) \leqq n$ if and only if every countable, locally finite Lebesgue cover of X has an open refinement of order $\leqq n+1$.*

PROOF. The proof is essentially the same proof as that of [9, Theorem 3.2].

THEOREM 3.3. *Let* $(X, \mathfrak{U})$ *be a uniform space. Then every countable Lebesgue cover of* X *has a countable locally finite Lebesgue refinement.*

PROOF. Let $\mathfrak{D}=\{D_1, D_2, \cdots\}$ be a Lebesgue cover of X. Then there exists $U\in\mathfrak{U}$ such that $\{U(x):x\in X\}$ refines $\mathfrak{D}$. Choose V and K symmetric in $\mathfrak{U}$ such that $K\subset K^4\subset V\subset V^2\subset U$ and define $F_i=\{x: V(x)\cap[X-D_i]=\varnothing\}$ for all i. As before $\mathfrak{F}=\{F_i:i=1, 2, \cdots\}$ is a Lebesgue cover of X. Now let

$$R_i = D_i - \bigcup_{j<i} [\mathrm{St}(F_j, \mathfrak{K})]^-, \quad \text{where} \quad \mathfrak{K} = \{K(x): x\in X\}.$$

Clearly $\mathfrak{R}=\{R_1, R_2, \cdots\}$ refines $\mathfrak{D}$ in a $1-1$ manner. We assert that $\mathfrak{R}$ is a locally finite Lebesgue cover of X.

(i) Let $x\in X$. Choose the smallest i such that $x\in[\mathrm{St}(F_i, \mathfrak{K})]^-$. Then $x\in D_i-\bigcup_{j<i}[\mathrm{St}(F_j, \mathfrak{K})]^-=R_i$ and hence R covers X. Also $\mathrm{St}(F_j, \mathfrak{K})\cap R_i=\varnothing$ for all $i>j$ so that $\mathfrak{R}$ is locally finite.

(ii) Let $x\in X$. Choose the smallest i such that $K(x)\cap[\mathrm{St}(F_i, \mathfrak{K})]^-\neq\varnothing$. Clearly $K(x)\subset X-\bigcup_{j<i}[\mathrm{St}(F_i, \mathfrak{K})]^-$. We claim that $K(x)\subset D_i$. Then $K(x)\subset R_i$ and hence $\mathfrak{R}$ is Lebesgue.

Let $y\in K(x)$. Since $K(x)$ can be open, there exists $r\in X$ such that $r\in K(x)\cap\mathrm{St}(F_i, \mathfrak{K})$. Thus there exist $s\in X$ and $t\in F_i$ such that $r\in K(s)$ and $t\in K(s)\cap F_i$. Therefore we have $t\in K(s)$, $s\in K(r)$, $r\in K(x)$, and $x\in K(y)$. Hence $(t, y)\in K^4\subset V$ so that $y\in V(t)$. By definition $t\in F_i$ implies that $V(t)\cap[X-D_i]=\varnothing$. Thus $y\in V(t)\subset D_i$ so that $K(x)\subset D_i$.

We now have a Lebesgue characterization for d_6.

THEOREM 3.4. *Let* $(X, \mathfrak{U})$ *be a paracompact uniform space. Then* $d_6(X, \mathfrak{U})\leqq n$ *if and only if every countable Lebesgue cover has an open refinement of order* $\leqq n+1$.

4. The uniformity dimension function d_7

DEFINITION 4.1. Let $(X, \mathfrak{U})$ be a uniform space. If $X=\varnothing$, then $d_7(X, \mathfrak{U})=-1$. Otherwise, $d_7(X, \mathfrak{U})\leqq n$ if $(X, \mathfrak{U})$ satisfies this condition.

(D_7) Given any collection of closed pairs $\{C_\alpha, C'_\alpha:\alpha\in A\}$ such that

(1) $\{C_\alpha, C'_\alpha:\alpha\in A\}$ are uniformly separated.
(2) $\{X-C'_\alpha:\alpha\in A\}$ is locally finite.

Then there exist closed sets B_α separating C_α and C'_α such that $\mathrm{ord}\{B_\alpha:\alpha\in A\}\leqq n$.

THEOREM 4.2. *Let* $(X, \mathcal{U})$ *be a paracompact uniform space. Then* $d_7(X, \mathcal{U}) \leqq n$ *if and only if every locally finite Lebesgue cover of* X *has a refinement of order* $\leqq n+1$.

PROOF. The proof is essentially the same as proof of [9, Theorem 4.2]. Paracompactness is now required so that [6, Theorem 1.3] is applicable.

5. **Conclusion.** The table in paragraph 1 is now complete for uniform spaces $(X, \mathcal{U})$. The Lebesgue characterizations are exactly the same as for metric spaces but complete normality is required for the dimension functions d_2, d_3 and paracompactness is required for d_6, d_7 and d_0.

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