SECONDARY COHOMOLOGY OPERATIONS AND COMPLEX VECTOR BUNDLES

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Secondary cohomology operations have proved very useful in recent years [2], [5], [6].

Here we show that certain secondary operations are associated with the divisibility by 2 of the Chern classes of complex vector bundles, and so we obtain a very simple method for evaluating them.

Thus let Φ_{4j} be the secondary operation based on the relation

$$\mathrm{Sq}^1\mathrm{Sq}^{4j} + (\mathrm{Sq}^2\mathrm{Sq}^1)\mathrm{Sq}^{4j-2} + \mathrm{Sq}^{4j}\mathrm{Sq}^1 = 0.$$

Let e be a generator of $H^2(CP^\infty, Z)$ and $\rho: H^*(\ , Z) \to H^*(\ , Z_2)$ the coefficient homomorphism.

THEOREM A. *Let k be divisible by 4 and suppose the $2j$th integral Chern class C_{2j} of $k\eta$ (η is the canonical line bundle over CP^∞) satisfies $C_{2j} = 2\theta$; then $\Phi_{4j}(\rho(e^k)) = \rho(e^k) \cup \rho(\theta)$ with zero indeterminacy.*

The total Chern class of $k\eta$ is $(1+e)^k$; hence

$$C_{2j} = \binom{k}{2j} e^{2j}$$

and we have for example

$$\Phi_{2^r}(\rho(e^{2^r})) = \rho(e^{2^r + 2^{r-1}}).$$

More generally

COROLLARY. *Let $i = 2^r a$ and $4j = 2^{r-1} b$ with a, b odd; then*

$$\binom{2\ a}{b\ -1} \equiv 1(2)$$

implies $\Phi_{4j}(\rho(e^i)) = \rho(e^{i+2j})$.

This last result includes as special cases the basic results in this direction of [1], [5], [6]. The proof of A turns out to be very easy and is given in §2. Further, our viewpoint simplifies the original proof of Hopf invariant one as given in [1] to the point where it may be even easier than the recent K-theory proofs of [3], [4], [7], so we include

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an outline proof in §3. Finally, in §4 we extend these results to secondary operations on RP^∞.

Theorem A is essentially 3.3.2 of [9] and the methods of proof are similar. A can also be obtained by using the results of [10] and S-duality [11, §4]. Maunder's approach allows one to evaluate certain higher order operations built up from the Φ_{4j} in CP^∞ as well. However our Theorem B does not seem to follow from [10] or [11], and the methods in §§2 and 4 also generalize to higher order operations (see [8]). Indeed our main object in presenting this paper was to help clarify [8]. Moreover, the viewpoint on higher order operations implicit in this note seems to be lacking in the literature.

2. **Theorem B and the proof of Theorem A.** Let E_n be the universal example for Φ_{4j} on an integral class of dimension n. Thus E_n is the fiber in the map

2.1
$$\begin{aligned}\theta = \mathrm{Sq}^{4j-2}(\iota) \times \mathrm{Sq}^{4j}(\iota) &: K(Z, n)\\ &\to K(Z_2, n + 4j - 2) \times K(Z_2, n + 4j)\end{aligned}$$

and we have the sequence of fibrations

2.2
$$\begin{aligned}F = K(Z_2, n + 4j - 3) \times K(Z_2, n + 4j - 1) &\xrightarrow{j} E_n \xrightarrow{\pi} K(Z, n)\\ &\xrightarrow{\theta} K(Z_2, n + 4j - 2) \times K(Z_2, n + 4j).\end{aligned}$$

Let $T(k)$ be the Thom space of the universal complex k-plane bundle, and denote the Thom class as $U \in H^{2k}(T(k), Z)$. From the map $(U): T(k) \to K(Z, 2k)$ and the fibration $\pi: E_{2k} \to K(Z, 2k)$ we have the induced fibration $(U)^\# E_{2k}$ over $T(k)$ and the diagram of fibrations:

$$\begin{tikzcd} (U)^\# E_{2k} \arrow[r, "(\tilde{U})"'] \arrow[d, "\tilde{\pi}"] & E_{2k} \arrow[d, "\pi"] \\ T(k) \arrow[r, "(U)"'] & K(Z, 2k) \end{tikzcd}$$

The following lemma is evident.

LEMMA 2.3. *Let $v \in H^*(E_{2k})$ be a universal example for Φ_{4j}. Let $\overline{U}$ be the Thom class of a complex k-bundle ξ over a space X; then*

$$\Phi_{4j}(\overline{U}) = \{L^*((\tilde{U})^* v)\}$$

where L runs over all liftings which make the following diagram com-

mute:

$$\begin{array}{ccc} & & (U)^{\#}E_{2k} \\ & L\nearrow & \downarrow \pi \\ T(\xi) & \longrightarrow & T(k) \end{array}$$

LEMMA 2.4. *In* $H^*(U^{\#}E_{2k}, Z)$ *there are classes* X, Y *so*
(i) $\tilde{\pi}^*(C_{2j-1}\cup U)=2X$
(ii) $\tilde{\pi}^*(C_{2j}\cup U)=2Y$
(iii) $\tilde{U}^*(\mu) = \rho(Y) + \mathrm{Sq}^2\rho(X) + \tilde{\pi}^*(\rho(P\cup U))$ *where* P *is a polynomial of degree* $4j$ *in the Chern classes* $C_1, \cdots, C_{2j-2}$,

PROOF. $H^{2k+4j}(U^{\#}E_{2k}, Z_2)=(\mathrm{im}\,\tilde{\pi}^*)\oplus(\mathrm{im}\ j^*)$, and also $j^*(\tilde{U}^*(v)) = \mathrm{Sq}^1(\iota_{2k+4j-1})+\mathrm{Sq}^2\mathrm{Sq}^1(\iota_{2k+4j-3})$; hence the lemma follows when we observe that if X, Y exist then they must restrict to $\mathrm{Sq}^1(\iota_{2k+4j-3})$, $\mathrm{Sq}^1(\iota_{2k+4j-1})$ respectively, since this is true in the universal example for division by two, namely the fibering

$$K(Z_2, n-1) \rightarrow K(Z, n) \xrightarrow{2(\iota)} K(Z, n).$$

On the other hand, to show the existence of X, Y is now completely direct.

Putting these results together we have

THEOREM B. *Let* B *be the Thom space of* ξ, *a complex* k*-bundle over a space* X *for which* $\rho(C_1)=0$. *Suppose there are integer classes* X, Y *so* $C_{2j-1}(\xi)=2X$, $C_{2j}(\xi)=2Y$; *then* $\Phi_{4j}(\rho U)$ *is defined in* $H^*(T, Z_2)$ *and modulo the total indeterminacy*

$$\Phi_{4j}(\rho U) = [\mathrm{Sq}^2\,\rho(X) + \rho(Y)] \cup \rho(U).$$

To complete the proof of B note that in the map

$$\lambda\colon \Sigma^{2k+4j-4}T(2j-2) \rightarrow T(k)$$

we have

$$\lambda^*\rho(P \cup U) = \Phi_{4j}(\sigma^{2k-4j-4}U_{2j-2}) = \sigma^{2k-4j-4}\Phi_{4j}(U_{2j-2})$$

and v can be chosen so this last is zero,[2] but since P is a polynomial in $C_1, \cdots, C_{2j-2}$ the fact that $\lambda^*(\rho P\cup U)=0$ implies $\rho(P\cup U)=0$ and B follows since $\mathrm{Sq}^2(\rho(U))=0$.

To complete the proof of A note that the Thom space of $k\eta$ over

[2] Φ_{4j} is defined on ρ of any integral $4j-3$ class, and thus its value on a $4j-4$ class a is a stable cohomology operation $\mathrm{Sq}^I(a)$. Then we choose a new representant for Φ, specifically we set $v'' = v+\mathrm{Sq}^I(\iota)$.

CP^∞ is CP^∞/CP^{k-1}. The theorem now follows from B and the evident map $p: CP^\infty \to CP^\infty/CP^{k-1}$ which collapses the $2k-1$ skeleton to the base point since $\rho(p^*T(k\eta)^*(U)) = \rho(e^k)$.

3. **Hopf invariant one.** $\mathfrak{A}(2)$ is the Steenrod algebra [12]. $\mathfrak{A}(2) \otimes \mathfrak{A}(2)$ becomes an $\mathfrak{A}(2)$ module when we set $\alpha(a\otimes b) = \alpha a \otimes b$, and $m: \mathfrak{A}(2)\otimes\mathfrak{A}(2) \to \mathfrak{A}(2)$ is given by $m(a\otimes b) = ab$.

Definition 3.1. An element $x = \sum a_i \otimes \mathrm{Sq}^{2^i} \in \mathfrak{A}(2)\otimes\mathfrak{A}(2)$ is a minimal relation if $m(x) = 0$.

Clearly the set of minimal relations is a module $\mathfrak{I}$ over $\mathfrak{A}(2)$ and a basic set of relations is any $\mathfrak{A}(2)$ basis for $\mathfrak{I}$, say $\{e_i : i \in I\}$ for some index set I. A relation among relations is an element $y = \sum a_i \otimes b_{i,j} \otimes \mathrm{Sq}^{2^j}$ in $\mathfrak{A}(2)\otimes\mathfrak{A}(2)\otimes\mathfrak{A}(2)$ so $(m\otimes 1)(y) = (1\otimes m)(y) = 0$.

Lemma 3.2. *Let $E_{n,i}$ be the two-stage Postnikov system obtained by killing* $\mathrm{Sq}^1(\iota)$, $\mathrm{Sq}^2(\iota)$, $\mathrm{Sq}^4(\iota) \cdots \mathrm{Sq}^{2^i}(\iota)$ *in* $H^*(K(Z_2, n), Z_2)$, $(n > 2^k)$; *then* $H^*(E_{n,i}, Z_2)$ *is isomorphic to the module of minimal relations (over* $\mathfrak{A}(2)$ *and with degree diminished by one) in dimension less than* $n + 2^{i+1}$. *In dimension* $n+2^{i+1}$ *there is also* $\mathrm{Sq}^{2^{i+1}}(\iota)$, *and if* y *is a relation among relations* $(\sum a_i \otimes b_{i,j} \otimes \mathrm{Sq}^{2^j})$ *then in* $H^*(E_{n,i}, Z_2)$ *we have* $\sum a_i \{b_{i,j} \otimes S_2^{2j}\} = \lambda(y)\mathrm{Sq}^{2^{i+1}}(\iota)$ *where* $\lambda = 0$ *or* 1 *and depends only on* y.

(The proof is a simple exercise with the Serre spectral sequence, the Steenrod algebra and the stability of the Steenrod operations.)

Lemma 3.3 (J. F. Adams). (a) *A basic set of minimal relations is given by a doubly indexed family* $R(i,j)$ *(of degree* $2^i + 2^j$*)* $0 \leqq i < j-1$ *or* $i = j$. (b) *There is a relation among relations of the form*

$$(\mathrm{Sq}^{2^i} + b)\{R(0, i)\} + \cdots$$

where b *is decomposable, for* $i \geqq 3$.

(The proof is an exercise in handling $\mathfrak{A}(2)$. Part (b) follows, for example, by proving $h_0 h_i^2$ is nonzero in $\mathrm{Ext}_{\mathfrak{A}(2)}(Z_2, Z_2)$ for $i \geqq 3$.)

Now consider the mapping $(\iota): E_{n,i} \to E_n$ where E_n is the universal example for Φ_{2^i}. The following is immediate.

Lemma 3.4. $(\iota)^*(v) = \{R(0, i)\} + \sum a_{j,k}\{R_{(j,k)}\}$ *where* $k < i$.

Now the basic theorem of [1] becomes

Theorem 3.5. *In* $H^*(E_{n,i}, Z_2)$ *we have* $(\mathrm{Sq}^{2^i} + b)\{R(0, i)\} + \cdots = \mathrm{Sq}^{2^{i+1}}(\iota)$ *for* $i \geqq 3$, *and thus there is no two cell complex with* $\mathrm{Sq}^{2^{i+1}}$ *nonzero,* $i \geqq 3$.

Proof. By A and 3.4 the only secondary operation among the

$\{R(j, k)\}$ $(k\leqq i)$ which is nonzero on $\rho(e^{2^i})$ is $\{R(0, i)\}$ (the $R(j, k)$ for $j>0$ all have odd degree, and $R(0, k)$ for $k<i$ are 0 by A). Hence $[(\mathrm{Sq}^{2^i}+b)\{R(0, i)\}+\cdots](\rho(e^{2^i}))=\mathrm{Sq}^{2^i}\{R(0, i)\}\rho(e^{2^i})=\mathrm{Sq}^{2^i}\rho(e^{2^i+2^{i-1}})$ $=\mathrm{Sq}^{2^{i+1}}(\rho(e^{2^i}))$ and the proof is complete.

4. **Secondary operations in** RP^∞. Consider again the results of §3. Φ_8 is defined and nonzero on $\rho(e^{8+16k})$ in $H^*(CP^\infty)$. But if $\pi: RP^\infty \to CP^\infty$ is the nontrivial map then $\Phi_8(\pi^*(\rho(e^{8+16k})))=0$ due to indeterminacy. On the other hand, from the universal relation $\mathrm{Sq}^{16} = \mathrm{Sq}^8\Phi_8+\cdots$ it follows that the set

$$\{\Phi_8(\pi^*(\rho(e^{8+16k}))),\ \{R(3, 3)\}(\pi^*(\rho(e^{8+16k}))),\ \cdots\}$$

cannot *all* vanish simultaneously. Thus this *set of operations*, even modulo indeterminacy, is not zero on the class of dimension $16+32k$ in $H^*(RP^\infty, Z_2)$.

There are similar results with Sq^{16} replaced by Sq^{2^i} $(i>4)$, and we have

THEOREM 4.1. *There is no map*

$$RP^{2^{i+1}(k+1)-1}_{2^i+2^{i+1}k} \to S^{2^i+2^{i+1}k}$$

inducing the nontrivial map in cohomology (RP^s_t *is* RP^s *with the* $t-1$ *skeleton collapsed to the base point*).

Actually we can sharpen this result. Consider, for example, the pair of relations

$$\mathrm{Sq}^1\mathrm{Sq}^{8k} + (\mathrm{Sq}^2\mathrm{Sq}^1)\mathrm{Sq}^{8k-2} + \mathrm{Sq}^{8k}\mathrm{Sq}^1 = 0,$$

$$\mathrm{Sq}^2\mathrm{Sq}^{8k} + \mathrm{Sq}^4\mathrm{Sq}^{8k-2} + \mathrm{Sq}^{8k}\mathrm{Sq}^2 + \mathrm{Sq}^{8k+1}\mathrm{Sq}^1 = 0.$$

Let G_n be the universal example for both operations, let $u\in H^*(G_n)$ represent Φ_{8k} and ω represent the second operation; then

$$j^*(\mathrm{Sq}^8(u) + (\mathrm{Sq}^7 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)\omega)$$

$$= (\mathrm{Sq}^9 + \mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)A + \mathrm{Sq}^{10}\mathrm{Sq}^1B + (\mathrm{Sq}^7 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)\mathrm{Sq}^{8k}C + (\mathrm{Sq}^8 + \mathrm{Sq}^7\mathrm{Sq}^1)\mathrm{Sq}^{8k}D \tag{4.2}$$

where D is the n-dimensional generator on the fiber, C is the $n+1$-dimensional generator, etc.

Let $\rho(e^t)$ satisfy $\Phi_{8k}(\rho(e^t))\neq 0$ in $H^*(CP^\infty)$; then there is a map $m: CP^\infty \to G_{2t}$, and we have $m^*(\iota)=\rho(e^t)$ $m^*(u)\neq 0$ and $m^*(\omega)=0$. Thus the same is true for $m\pi: RP^\infty \to G_{2t}$. Moreover, any two liftings

$$\begin{tikzcd} & G_{2t} \arrow[d] \\ RP^\infty \arrow[ur, "m,\, \bar{m}^u"] \arrow[r, "(\pi^*\rho(e^t))"'] & K(Z_2, 2t) \end{tikzcd}$$

differ by a map into the fiber F. Thus if there were a lifting $\bar{m}$ so $\bar{m}^*(u) = \bar{m}^*(\omega) = 0$ there would be a map $r: RP^\infty \to F$ and

4.3. $$r^*(j^*(u)) \neq 0 \qquad \text{while } r^*(j^*(\omega)) = 0.$$

THEOREM 4.4. *Suppose $k = 1+4s$. Then $(\Phi_{8k};\ \omega)$ on $(\pi^*\rho(e^{8(1+2\lambda)}))$ cannot both vanish simultaneously if*

$$\binom{\lambda}{s} \neq 0.$$

PROOF. Under these assumptions

(1) $$\mathrm{Sq}^{8k}\rho(e^{8(1+2\lambda)+\epsilon}) = 0 \qquad (\epsilon = 0, 1),$$

(2) $$\Phi_{8k}(\rho(e^{8(1+2\lambda)})) \neq 0,$$

(3) $$\mathrm{Sq}^8\Phi_{8k}\rho(e^{8(1+2\lambda)}) \neq 0.$$

Suppose now there were a map $r: RP^\infty \to F$ satisfying 4.3; then from 4.2

$$r^*(\mathrm{Sq}^8 j^*(u) + \mathrm{Sq}^2 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 j^*(\omega)) = 0,$$

but this is impossible since $\mathrm{Sq}^8(r^*j^*(u)) \neq 0$ and by assumption $r^*j^*(\omega) = 0$.

Similar results can be obtained for Φ_{8k+4} but we need three operations based on

$$\begin{aligned} \mathrm{Sq}^1\mathrm{Sq}^{8k+4} + (\mathrm{Sq}^2\mathrm{Sq}^1)\mathrm{Sq}^{8k+2} + \mathrm{Sq}^{8k+4}\mathrm{Sq}^1 &= 0, \\ \mathrm{Sq}^4\mathrm{Sq}^{8k+2} + \mathrm{Sq}^{8k+4}\mathrm{Sq}^2 &= 0, \\ \mathrm{Sq}^4\mathrm{Sq}^{8k+4} + \mathrm{Sq}^{8k+6}\mathrm{Sq}^2 + \mathrm{Sq}^{8k+7}\mathrm{Sq}^1 &= 0, \end{aligned}$$

and if u represents the first, ω the second, and χ the third, we look at

$$j^*(\mathrm{Sq}^8(u) + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1(\bar{\omega}) + \mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2(\chi))$$

on the fiber. The reader can easily supply details and further generalizations.

REMARK. One could, of course, verify directly (as in [2], [5], [6]) that these operations do not vanish simultaneously, since we know how to evaluate the indeterminacy. However, the method given here generalizes to higher order operations as in §3.4 of [8].

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