REMARKS ON COMMUTING INVOLUTIONS

LAWRENCE CONLON

In [**3**, p. 293] R. Hermann poses the following problem (without the restriction that G be simple).

(A) Given s_1 and s_2 nontrivial involutive automorphisms of a compact simple Lie group G, find $x \in G$ such that $\mathrm{Ad}(x)s_1\mathrm{Ad}(x)^{-1}$ commutes with s_2.

We wish to discuss the existence of solutions for (A). Without real loss of generality we assume G simply connected. The respective fixed point groups of s_1 and s_2 are closed connected subgroups K_1 and K_2 of G, and K_1 acts from the left on G/K_2. In both [**1**] and [**3**] it is shown that there is a flat geodesically imbedded torus $T \subset G/K_2$ which meets orthogonally every K_1-orbit. Furthermore, if the decompositions of the Lie algebra $\mathfrak{g}$ of G into $+1$ and -1 eigenspaces are given respectively by

$$\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{m}_1$$

$$\mathfrak{g} = \mathfrak{k}_2 \oplus \mathfrak{m}_2$$

then T has as universal covering a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{m}_1 \cap \mathfrak{m}_2$. Indeed, T may be so chosen that, under the standard imbedding $G/K_2 \subset G$ (given by $zK_2 \to s_2(z)z^{-1}$), it becomes identified with $\exp(\mathfrak{t})$. A complete description of the singular set in T is given in [**1**] by a finite system $\mathfrak{A}$ (called an "affine root system") of affine functionals defined on $\mathfrak{t}$.

(B) THEOREM. (A) *has a solution* $x \in G$ *if and only if some translation in* $\mathfrak{t}$ *carries* $\mathfrak{A}$ *to a system* $\mathfrak{A}'$ *such that* $\omega(0) = 0$ *or* $\frac{1}{2}$, *for all* $\omega \in \mathfrak{A}'$.

PROOF. By [**1**, pp. 233–234] translations in $\mathfrak{t}$ correspond to replacing K_1 by $\mathrm{Ad}(x)K_1$ for suitable $x \in T$, hence to replacing s_1 by $\mathrm{Ad}(x)s_1\mathrm{Ad}(x)^{-1} = s_1'$. If such a translation produces a system $\mathfrak{A}'$ in which $\omega(0) = 0$ or $\frac{1}{2}$, for every $\omega \in \mathfrak{A}'$, then by [**1**, Proposition S-3], $s_1's_2$ is an involution. It follows that $s_1's_2 = s_2s_1'$; hence $x \in T \subset G$ solves (A).

For the converse, let $x \in G$ be such that $\mathrm{Ad}(x)s_1\mathrm{Ad}(x)^{-1}$ commutes with s_2. We must show that x can be chosen as an element of T, in which case [**1**] will show that the affine root system translates to a system with all constant terms 0 or $\frac{1}{2}$. From Proposition 1.4 of [**1**] we easily see that $G = K_2TK_1$. Write $x = x_2yx_1$ for suitable $y \in T$,

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$x_i \in K_i$, $i=1, 2$. Then

$$\mathrm{Ad}(x)s_1\mathrm{Ad}(x)^{-1} = \mathrm{Ad}(x_2y)s_1\mathrm{Ad}(x_2y)^{-1}$$

and this commutes with s_2. Therefore, $\mathrm{Ad}(y)s_1\mathrm{Ad}(y)^{-1}$ commutes with $\mathrm{Ad}(x_2)^{-1}s_2\mathrm{Ad}(x_2)=s_2$. q.e.d.

The classification [2] has shown via (B) that (A) fails to have a solution in the cases equivalent to the following.

	G	K_1	K_2
(1)	$\mathrm{SU}(2q)$	$\mathrm{Sp}(q)$	$U(2q-1)$
(2)	$\mathrm{SU}(2r+2q)$ $(q>r+1)$	$\mathrm{Sp}(r+q)$	$S(U(2q-1)\times U(2r+1))$
(3)	$\mathrm{Spin}(2q)$	$U(q)$	$\mathrm{Spin}(2q-1)$
(4)	$\mathrm{Spin}(2r+2q+2)$ $(q>r)$	$U(r+q+1)$	$\mathrm{Spin}(2r+1)\times_{Z_2}\mathrm{Spin}(2q+1)$
(5)	$\mathrm{Spin}(8)$	$\mathrm{Spin}(7)$	$\omega(\mathrm{Spin}(7))$
(6)	$\mathrm{Spin}(8)$	$\mathrm{Spin}(7)$	$\omega(\mathrm{Spin}(3)\times_{Z_2}\mathrm{Spin}(5))$
(7)	$\mathrm{Spin}(8)$	$\mathrm{Spin}(3)\times_{Z_2}\mathrm{Spin}(5)$	$\omega(\mathrm{Spin}(3)\times_{Z_2}\mathrm{Spin}(5))$

Here ω is the triality automorphism of Spin(8) and the various subgroups are standardly imbedded. In all other cases (A) has a solution.

Hermann shows [3, Proposition 2.1] that K_1 has a totally geodesic orbit in G/K_2 if and only if (A) can be solved. Actually, as the following proposition shows, the cases in which K_1 is transitive on G/K_2 constitute technical counterexamples to Hermann's result (and were implicitly excluded in his proof).

(C) PROPOSITION. *If K_1 is transitive on G/K_2, then* (A) *cannot be solved.*

PROOF. Suppose the action transitive. If (A) has a solution $x \in G$, then $\mathrm{Ad}(x)K_1$ is also transitive on G/K_2, so we may as well assume $s_1s_2 = s_2s_1$. Necessarily $K_1K_2 = G$; hence $\mathfrak{k}_1 + \mathfrak{k}_2 = \mathfrak{g}$ and $\mathfrak{m}_1 \cap \mathfrak{m}_2 = 0$. s_1s_2 is an involutive automorphism of $\mathfrak{g}$ with fixed point algebra $\mathfrak{k}_1 \cap \mathfrak{k}_2$ and -1 eigenspace $\mathfrak{m}_1 \oplus \mathfrak{m}_2$. Thus $\mathfrak{g} = \mathfrak{k}_1 \cap \mathfrak{k}_2 \oplus (\mathfrak{m}_1 \oplus \mathfrak{m}_2)$ is the decomposition corresponding to some symmetric space. But $\mathrm{ad}_{\mathfrak{m}_1 \oplus \mathfrak{m}_2}(\mathfrak{k}_1 \cap \mathfrak{k}_2)$ is a reducible representation, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ being invariant subspaces. By standard theory of symmetric spaces, this contradicts the fact that $\mathfrak{g}$ is simple. q.e.d.

The transitive cases in the above table are precisely (1), (3), (5), and (6) (cf. [2], [4]).

Finally, we remark that a more careful study of the affine root system $\mathfrak{A}$ permits a complete description of the totally geodesic K_1-orbits in G/K_2 (cf. [2]).

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ST. LOUIS UNIVERSITY