POSITIVE OPERATORS ON $C(X)$

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ABSTRACT. *Conditions for the existence of finite and σ finite invariant measures for a Markov operator on $C(X)$ are studied.*

Let X be a locally compact Hausdorff space and Σ be the collection of Baire sets. Let P be an operator on $C(X)$ which satisfies

(1) $\|P\|=1$,

(2) $P1=1$.

Now if $f \geqq 0$ then $Pf \geqq 0$, because we may assume that $0 \leqq f \leqq 1$ and then $P(1-f)=1-Pf \leqq 1$.

We shall also assume:

(3) *If μ is a countably additive measure then so is $P^*\mu$.*

By a measure we mean a positive finite measure, unless otherwise stated. Every countably additive measure is regular since it is defined on Baire sets.

It is enough to assume (3) only for $\mu=\delta_x$, $x \in X$, since then $P(x, A) = P^*\delta_x(A)$ is a probability measure for a fixed x and if $A \in \Sigma$, $P(\cdot, A)$ is Σ measurable: If $A \in \Sigma$ is compact then there exists a sequence of continuous functions $0 \leqq f_n \leqq 1$, $f_n \downarrow 1_A$ (where $1_A(x)=0$ if $x \notin A$ and 1 if $x \in A$). Thus $P(x, A)=\lim_n P^*\delta_x(f_n)=\lim_n Pf_n(x)$ and is measurable. Now the collection of $A \in \Sigma$ such that $P(\cdot, A)$ is measurable is a σ field and thus all of Σ. Clearly $P^*\mu(A)=\int P(x, A)\mu(dx)$ is a countably additive measure whenever μ is. Also $Pf(x)=\int P(x, dy)f(y)$ and if $0 \leqq f_n \leqq 1$ and $f_n \downarrow f$ then $Pf_n(x) \rightarrow Pf(x)$. This last property almost implies (3).

THEOREM 1. *Let X be a locally compact Hausdorff space and $X=\bigcup C_n$ where C_n are compact Baire sets. If the operator P satisfies* (1), (2) *and* (3′):

(3′) *If $0 \leqq f_n \leqq 1$, $f_n \downarrow f$ and $f_n, f \in C(X)$ then $Pf_n(x) \rightarrow Pf(x)$.*

Then condition (3) *holds.*

PROOF. The operator P^* is defined on finitely additive regular measures, by [**1**, Theorem IV.6.2]. Let $0 \leqq \mu$ be countably additive and put $P^*\mu=\nu_0+\nu_1$ as in [**4**, p. 52]; namely, ν_1 is countably additive and ν_0 is purely additive, i.e. if $0 \leqq \lambda \leqq \nu_0$ and λ is countably additive then $\lambda=0$. Now if $A \in \Sigma$ is compact the restriction of ν_0 to A is countably additive by [**1**, Theorem III. 5.13] and thus $\nu_0(A)=0$. Hence

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$\nu_0(f)=0$ for every continuous function with compact support. There exists a sequence of functions f_n, $0 \leqq f_n \leqq 1$, f_n continuous with compact support, $f_n \geqq 1_{C_n}$. We may assume that the sequence is monotone since we may replace it by the sequence $\max(f_1, \cdots, f_n)$. Thus $f_n \uparrow 1$ and $\nu_1(X)=\lim \nu_1(f_n)=\lim P^*\mu(f_n)=\lim \mu(Pf_n)$ and by (3′) the limit is $\mu(X)$. Thus $\nu_0(X)=0$ and $\nu_0=0$.

REMARKS. If X is compact then (3′) follows from Dini's Theorem. We do not know if (3′) suffices without assuming that X is the union of countably many compact sets.

In [2, Theorem 1] the problem of existence of a finite measure was studied. This result can be improved.

THEOREM 2. *Assume* (1), (2) *and* (3) *and let A be a compact Baire set. Then either*

(a) $\sup_x (1/N) \sum_{n=0}^{N-1} P^n 1_A(x) \to 0$ *or*

(b) *there exists a countably additive invariant measure that does not vanish on A.*

PROOF. Assume that (a) is false. Let $\delta>0$ and N_i be an infinite sequence such that

$$A_{N_i} = \left\{x: \frac{1}{N_i} \sum_{n=0}^{N_i-1} P^n 1_A(x) > \delta\right\}$$

are not empty. Put $\mu_i=\delta_{x_i}$, $x_i \in A_{N_i}$ and

$$\lambda_i = \frac{1}{N_i} \sum_{n=0}^{N_i-1} P^{*n}\mu_i.$$

Then λ_i is countably additive $0 \leqq \lambda_i$, $\lambda_i(X)=1$ and $\lambda_i(A) \geqq \delta$. Now any weak $*$ limit of λ_i is an invariant measure that does not vanish on A.

To conclude this paper let us consider the sufficient condition, for the existence of a σ finite invariant measure studied in [3]. Put $T_A f(x) = 1_A(x) f(x)$ then, by [3], an invariant σ finite measure exists if for some Baire set A, whose complement is compact,

(4) $(T_A P T_A)^n 1(x) \to 0$ *for every* $x \in X$, *as* $n \to \infty$.

Also the invariant measure is finite on every compact set.

Let $\mathfrak{U}$ be the collection of all open Baire sets whose closure is compact.

LEMMA 1. *Condition* (4) *holds for every set A whose complement B is the closure of a set in $\mathfrak{U}$ if and only if condition* (4) *holds for every set A_1 whose complement B_1 belongs to $\mathfrak{U}$.*

PROOF. Let B be the closure of B_1 in $\mathfrak{U}$ then if A, A_1 are complements, respectively, of B and B_1, $A \subset A_1$, and so $(T_A P T_A)^n 1 \leqq (T_{A_1} P T_{A_1})^n 1$ and if the right-hand side converges to zero so does the left-hand side. Conversely let $B_1 \in \mathfrak{U}$; find an open set $B_0 \in \mathfrak{U}$ with $B_0 \subset \mathrm{cl}(B_0) \subset B_1$. Put A as the complement of $\mathrm{cl}(B_0)$ and A_1 the complement of B_1 then $A_1 \subset A$ and $(T_{A_1} P T_{A_1})^n 1 \leqq (T_A P T_A)^n 1$.

As in [**3**, Lemma 2] it is easy to show that if (4) holds for every set A, whose complement is in $\mathfrak{U}$ then $\sum_{n=1}^{\infty} P^n 1_B(x) > 0$ for every $x \in X$ and $B \in \mathfrak{U}$. Thus if λ is any σ finite countably additive invariant measure then $\lambda(B) > 0$ for *every* $B \in \mathfrak{U}$.

THEOREM 3. *Let P satisfy* (1), (2) *and* (3). *The operator P satisfies* (4), *for every complement of a set in $\mathfrak{U}$, if and only if*

(5) *If g is upper semicontinuous, $0 \leqq g \leqq 1$, and $Pg \geqq g$, then g is a constant.*

PROOF. If condition (4) fails for some A whose complement $B \in \mathfrak{U}$, let α be a continuous function with $0 \leqq \alpha \leqq 1$, $\alpha \geqq 1_A$ and $\alpha = 0$ on some set C, $C \in \mathfrak{U}$. Then $(\alpha P \alpha)^n 1 \downarrow g$ where $g = 0$ on C, g is not identically zero. Now $g = \alpha P \alpha g \leqq P(\alpha g) \leqq Pg$ and g is upper semicontinuous which contradicts (5). Conversely, if g is upper semicontinuous and $Pg \geqq g$ find some $0 < a < 1$ so that $\{x : g(x) < a\}$ is an open set which is neither ϕ nor X. This is possible since g is not a constant. Put $g_1 = \max(g - a, 0)$ then g_1 is again upper semicontinuous, $Pg_1 \geqq g_1$ and $\{x : g_1(x) = 0\}$ contains an open set, V, in $\mathfrak{U}$. Let A be the complement of V then $(T_A P T_A) g_1 = T_A P g_1 \geqq T_A g_1 = g_1$ so $(T_A P T_A)^n 1 \geqq (T_A P T_A)^n g_1 \geqq g_1$ contradicting (4).

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