ON CENTRALIZERS OF INVOLUTIONS

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1. Introduction. The main purpose of this paper is to establish sufficient conditions for a group of even order to contain a normal elementary Abelian 2-subgroup of order at most 4 (Theorem 1). As a consequence it is shown that PSL(2, 5) is the only simple group which contains an involution x with the following property: the Sylow 2-subgroup of the centralizer C of x in G is a noncyclic group of order 4 which is normal in C (Theorem 3).

Several corollaries are derived from Theorem 1. In particular, a direct proof is given of the fact that PSL(2, 5) is the only group which has no normal 2-complement, no normal elementary Abelian 2-subgroups of order less than 8 and which contains an involution with an elementary Abelian centralizer of order 4 (Theorem 2).

If G is a group, $x \in G$ and T is a subset of G, $C_G(x)$, $\mathrm{Cl}_G(x)$, $I(T)$, $o(T)$, $o(x)$, $\langle T \rangle$, $T^\#$, $Z(G)$ and $K(G)$ denote respectively: the centralizer of x in G, the conjugate class of x in G, the set of involutions in T, the number of elements in T, the order of x, the group generated by T, $T-\{1\}$, the center of G and the largest normal subgroup of G of odd order. If P is a p-group then $\Omega_1(P)$ is the subgroup of P generated by elements of P of order p.

From now on G will be a group of even order, x a fixed involution of G, $K=K(G)$, $C=C_G(x)$, $I=I(C_G(x))$, $\mathrm{Cl}(x)=\mathrm{Cl}_G(x)$, and S a fixed Sylow 2-subgroup of G containing x such that $S_0=S\cap C=$ Sylow 2-subgroup of C. We are ready to state the results.

THEOREM 1. *Suppose that there exists $y \in I-\mathrm{Cl}(x)$ such that*

$$(*) \qquad C_G(u) \cap \mathrm{Cl}_G(y) \subset C_G(y)$$

for all $u \in I$. Then $\langle \mathrm{Cl}_G(y) \rangle$ is a proper elementary Abelian normal 2-subgroup of G.

If, in addition, $I \cap \langle \mathrm{Cl}_G(y) \rangle = \{y\}$, then $o(\langle \mathrm{Cl}_G(y) \rangle) \leqq 4$.

COROLLARY 1. *Suppose that the following conditions hold:*

(a) *$I=I(C_G(u))$ for all $u \in \mathrm{Cl}(x) \cap I$;*

(b) *$I(C_G(y))=I(C_G(z))$ for all y, $z \in I-\mathrm{Cl}(x)$. Then one of the following statements holds.*

(i) *G has one class of involutions and $\langle I \rangle$ is an elementary Abelian normal 2-subgroup of C.*

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(ii) *G has at least two classes of involutions and it contains a proper elementary Abelian normal 2-subgroup.*

COROLLARY 2. *Suppose that $o(I) \leqq 3$. Then one of the following statements holds.*

(i) *$S_0 = S$, x is the only involution in S and $\langle x \rangle K$ is a normal subgroup of G.*

(ii) *$S_0 = S$, S contains exactly 3 involutions and $\langle x \rangle K$ is a proper normal subgroup of G.*

(iii) *$S_0 = S$, G has one conjugate class of involutions.*

(iv) *G has at least 2 classes of involutions and it contains a normal elementary Abelian subgroup of order at most 4.*

Corollary 2 immediately yields

COROLLARY 3. *Suppose that $o(I) \leqq 3$ and G is simple. Then $S = S_0$ and G has only one conjugate class of involutions.*

In case that C is elementary Abelian of order 4 we get the following

THEOREM 2. *Suppose that $C = \{1, x, y, xy\}$ is elementary Abelian and G has neither a normal 2-complement nor a normal elementary Abelian 2-subgroup of order less than 8. Then $G \cong \mathrm{PSL}(2, 5)$.*

The following corollary is an easy consequence of Theorem 2, the results of Suzuki in [6] and the results of Feit and Thompson in [2].

COROLLARY 4. *Let G be a finite noncyclic simple group containing an element w such that $o(C_G(w)) \leqq 4$. Then G is isomorphic to one of the following groups*: $\mathrm{PSL}(2, 5)$, $\mathrm{PSL}(2, 7)$, A_6 *and* A_7.

Our final theorem requires the deep results of Gorenstein and Walter [5] with respect to groups with a dihedral Sylow subgroup of order 4.

THEOREM 3. *Suppose that $S_0 = \{1, x, y, xy\}$ is elementary Abelian, S_0 is normal in C and G is simple. Then $G \cong \mathrm{PSL}(2, 5)$.*

The proof of Theorem 1 utilizes the following lemma, which is of independent interest.

LEMMA. *Let U be a subgroup of the group H and let w be an involution of H which normalizes U leaving fixed exactly two elements of U, 1 and y. Let V be a normal, w-invariant noncyclic elementary Abelian subgroup of U containing y. Then V is a Sylow 2-subgroup of U, $o(V) = 4$, and U/V is Abelian.*

2. **Proof of the Lemma, Theorem 1 and Corollary 1.** We begin with the proof of the Lemma. Obviously y is an involution. First assume that $o(V)=4$, $V=\{1, y, z, yz\}$; then $z^w=yz$. Suppose that U/V is not an Abelian group of odd order. Then w fixes an element of $(U/V)^\#$, say uV. Thus one of the following holds:

$$\begin{aligned} u^w &= uy \quad \text{and} \quad & u = u^{w^2} &= u \\ &= uz & &= uy \\ &= uyz & &= uy. \end{aligned}$$

Hence we must have $u^w=uy$; but then $(uz)^w=(uy)(yz)=uz$ a contradiction. Thus U/V is an Abelian group of odd order. If $o(V)>4$, then w fixes an element of $(V/\langle y\rangle)^\#$, say $z\langle y\rangle$, and $V_0=\langle z, y\rangle$ is a normal, w-invariant, elementary Abelian subgroup of V containing y, $o(V_0)=4$, and by the first part $V=V_0$, a contradiction. The proof of the Lemma is complete.

To prove Theorem 1, suppose first that $\mathrm{Cl}_G(y)\not\subset C_G(y)$ and let $t\in\mathrm{Cl}_G(y)-C_G(y)$. By a result of Brauer and Fowler [**1**, p. 572], there exists $w\in I(G)$ such that $w\in I(C_G(x))\cap C_G(t)\subset I$. Hence by (*) $t\in C_G(w)\cap\mathrm{Cl}_G(y)\subset C_G(y)$ a contradiction. It follows that $\mathrm{Cl}_G(y)\subset C_G(y)$ and $\langle\mathrm{Cl}_G(y)\rangle=H$ is a normal subgroup of G contained in $C_G(y)$. If $C_G(y)=G$, then $H=\langle y\rangle\neq G$ and the theorem follows. If $C_G(y)\neq G$, then H is a proper normal subgroup of G and obviously $y\in\Omega_1(P)\lhd G$ where P is the Sylow 2-subgroup of $Z(H)$. Hence $\mathrm{Cl}_G(y)\subset\Omega_1(P)$ and H is elementary Abelian. Finally suppose that $o(H)\geqq 8$ and $I\cap H=\{y\}$. Then x leaves only y and 1 fixed in H and by the Lemma $o(H)=4$, a contradiction. Thus $o(H)\leqq 4$ and the proof of Theorem 1 is complete.

It remains to prove Corollary 1. If $I\subset\mathrm{Cl}(x)$, then each element of I belongs to the center of some Sylow 2-subgroup of G and therefore G has one class of involutions. By (a), $\langle I\rangle$ is an elementary Abelian normal 2-subgroup of C and (i) holds. Suppose finally that $I\not\subset\mathrm{Cl}(x)$ and let $y\in I-\mathrm{Cl}(x)$. It follows from (b) that the elements of $I-\mathrm{Cl}(x)$ commute with each other. Thus for all $u\in I\cap\mathrm{Cl}(x)$,

$$C_G(u)\cap\mathrm{Cl}_G(y) = I\cap\mathrm{Cl}_G(y)\subset C_G(y),$$

and for all $u\in I-\mathrm{Cl}(x)$,

$$C_G(u)\cap\mathrm{Cl}_G(y) = I(C_G(y))\cap\mathrm{Cl}_G(y)\subset C_G(y).$$

It follows then by Theorem 1 that G has a proper normal elementary Abelian 2-subgroup.

3. **Proof of Theorem 2 and Corollaries 2 and 4.** We begin with Corollary 2. If $o(I)=1$, then $S_0=S$, x is the only involution in S and by [**3**], $\langle x\rangle K$ is a normal subgroup of G, as described in (i). As $o(I)\neq 2$, let $o(I)=3$, $I=\{x, y, xy\}$. If no element of I is conjugate to x in G, then $N_S(S_0)=S_0$, $S=S_0$, and by [**3**] $\langle x\rangle K\lhd G$. Since $o(I)=3$, $\langle x\rangle K\neq G$ and (ii) holds. If all the elements of I are conjugate in G, then again $S_0=S$ and (iii) holds. Suppose finally that x is conjugate to xy in G, but not to y. Then $I(C_G(xy))=I$ and by Corollary 1, $\langle \mathrm{Cl}_G(y)\rangle$ is a normal elementary Abelian 2-subgroup of G. Hence, as either $\langle \mathrm{Cl}_G(y)\rangle=\langle y\rangle$ or $\mathrm{Cl}_G(y)$ contains an element which does not commute with x, $I\cap\langle \mathrm{Cl}_G(y)\rangle=\{y\}$ and by Theorem 1, $o(\langle \mathrm{Cl}_G(y)\rangle)\leqq 4$, so that (iv) holds. This completes the proof of Corollary 2.

We continue with Theorem 2. If $C=S$, then by Lemma 15.2.4 of [**4**], G has only one class of involutions and $N=N_G(C)\cong\mathrm{PSL}(2, 3)$. Thus C contains the centralizer of each of its nonunit elements and by Theorem 9.3.2 in [**4**], due to Suzuki, G is a Zassenhaus group of degree 5 with N the subgroup fixing a letter. Thus N is a Frobenius group with complement of order $e=3$ and kernel of order $n=4$. Since e is odd and $e=n-1$, it follows from Theorems 13.3.5 and 13.1.1 in [**4**], due to Zassenhaus, that $G\cong\mathrm{PSL}(2, 4)\cong\mathrm{PSL}(2, 5)$. Next assume that $C\neq S$ and let $y\in C\cap Z(S)$. As $N_S(C)\neq C$, xy is conjugate to x in G and $C_G(xy)=C$. Since y is not conjugate to x in G, it follows from Theorem 1 that $\langle \mathrm{Cl}_G(y)\rangle$ is a normal elementary Abelian 2-subgroup of G. As before $I\cap\langle \mathrm{Cl}_G(y)\rangle=\{y\}$, and it follows by Theorem 1 that $o(\langle \mathrm{Cl}_G(y)\rangle)\leqq 4$ in contradiction to our assumptions. The proof is complete.

It remains to prove Corollary 4. If $o(C_G(w))=2$, then G is not simple. If $o(C_G(w))=3$, then by [**2**], G is isomorphic either to $\mathrm{PSL}(2, 5)$ or to $\mathrm{PSL}(2, 7)$. If $o(C_G(w))=4$ and $o(w)=4$, then by [**6**], G is isomorphic to one of the groups $\mathrm{PSL}(2, 7)$, A_6 and A_7. If, finally, $o(C_G(w))=4$ and $o(w)=2$, then by Theorem 2, $G\cong\mathrm{PSL}(2, 5)$.

4. **Proof of Theorem 3.** If $S=S_0$, then by [**5**], $G\cong\mathrm{PSL}(2, q)$, $q>3$. If q is even, then $G\cong\mathrm{PSL}(2, 4)\cong\mathrm{PSL}(2, 5)$. If q is odd, then the centralizer C of an involution of G is a dihedral group of order $q+\epsilon$, $\epsilon=\pm 1$. For S to be normal in C, $q+\epsilon=4$ and $q=5$. Thus again $G\cong\mathrm{PSL}(2, 5)$. Suppose next that $S_0\neq S$, $\{y\}=Z(S)\cap S_0^{\#}$. Then $N_S(S_0)\neq S_0$, xy is conjugate to x in G and S_0 is the normal Sylow 2-subgroup of $C_G(xy)$. As y is not conjugate to x in G, it follows from Corollary 1 that G contains a proper, nontrivial, normal subgroup, in contradiction to the simplicity of G. The proof is complete.

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