AN INEQUALITY SIMILAR TO OPIAL'S INEQUALITY

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In a recent paper [5], Willet gave a generalization of Opial's inequality [4]: *If* $y \in C^{(n)}[a, b]$ *with* $y^{(i)}(a) = 0$ *for* $i = 0, 1, \cdots, n-1$, *then*

$$\int_a^b |yy^{(n)}|\, dx \leqq \frac{1}{2}(b-a)^n \int_a^b |y^{(n)}|^2\, dx. \tag{1}$$

(He employs (1) to prove uniqueness of the initial value problem for linear differential equations of order n.) Also there are generalizations of Opial's original inequality in other directions. (See, for instance, Calvert [2] and Yang [6].)

The purpose of this note is to obtain a sharper version of (1) and other generalizations.

THEOREM 1. *Let* $y \in C^{(n-1)}[a, b]$ *be such that* $y^{(i)}(a) = 0$ *for* $i = 0, 1, \cdots, n-1$ *where* $n \geqq 1$. *Let* $y^{(n-1)}$ *be absolutely continuous and* $\int_a^b |y^{(n)}|^2 dx < \infty$. *Then,*

$$\int_a^b |y(x)y^{(n)}(x)|\, dx \leqq K(b-a)^n \int_a^b |y^{(n)}(x)|^2\, dx, \tag{2}$$

where

$$K = \frac{1}{2(n!)}\left(\frac{n}{2n-1}\right)^{1/2}. \tag{3}$$

PROOF. In view of the assumptions on y, for any x in $[a, b]$

$$y(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} y^{(n)}(t)\, dt.$$

Therefore,

$$|y(x)y^{(n)}(x)| \leqq \frac{|y^{(n)}(x)|}{(n-1)!}\int_a^x (x-t)^{n-1}|y^{(n)}(t)|\, dt,$$

and, by Schwarz inequality,

$$|y(x)y^{(n)}(x)| \leqq \frac{|y^{(n)}(x)|}{(n-1)!}\,\frac{(x-a)^{n-1/2}}{(2n-1)^{1/2}}\left(\int_a^x |y^{(n)}(t)|^2 dt\right)^{1/2}.$$

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Thus, integrating from a to b and applying Schwarz inequality to the right-hand side again,

$$\int_a^b |y(x)y^{(n)}(x)|\,dx \leqq \frac{1}{(n-1)!(2n-1)^{1/2}}\left(\int_a^b (x-a)^{2(n)-1}dx\right)^{1/2} \cdot\left(\int_a^b |y^{(n)}(x)|^2\left(\int_a^x |y^{(n)}(t)^2 dt\right)dx\right)^{1/2}.$$

The above is, in fact, (2). This completes the proof.

REMARK. For $n=1$, (2) is Opial's original result. For $n\geqq 2$, (2) is evidently sharper than (1).

COROLLARY. *Equality holds in* (2) *if and only if* $n=1$ *and* $y^{(n)}(x)=\text{const.}$

PROOF. The "only if" part needs proof. Now, equality holds in (2) only if it holds in all intermediate steps. Therefore, it is necessary that

$$y^{(n)}(t) = C(x)(x-t)^{n-1}, \quad \text{for each } x, \quad \text{and } a \leqq t \leqq x.$$

Thus, unless $n=1$, $y^{(n)}(t)=0$, and if $n=1$, $y^{(n)}(t)=\text{const}$. Hence the assertion.

REMARK. *If* $y\in C[a,b]$ *satisfy that* $y', \cdots, y^{(n-1)}$ *are piecewise continuous,* $y^{(n-1)}$ *is absolutely continuous with* $\int_a^b |y^{(n)}|^2dx<\infty$, $y^{(i)}(a)=y^{(i)}(b)=0$ *for* $i=0,\cdots,n-1$, *then*

$$\int_a^b |y(x)y^{(n)}(x)|\,dx \leqq K\left(\frac{b-a}{2}\right)^n \int_a^b |y^{(n)}(x)|^2\,dx, \tag{4}$$

K as given in (3). *Equality holds in* (4) *iff* $n=1$ *and*

$$\begin{aligned} y(x) &= c(x-a)^n, & a \leqq x \leqq (a+b)/2, \\ &= c(b-x)^n, & (a+b)/2 \leqq x \leqq b. \end{aligned}$$

This is immediate on using (2) once on $[a, (a+b)/2]$ and again on $[(a+b)/2, b]$, where on the latter interval

$$y(x) = \frac{(-1)^n}{(n-1)!}\int_x^b (t-x)^{n-1}y^{(n)}(t)dt.$$

Following is a generalization of Theorem 1.

THEOREM 2. *Let p, q be both positive satisfying $p+q>1$. Let y be as in Theorem* 1 *and let* $\int_a^b |y^{(n)}|^{p+q}dx<\infty$. *Then,*

$$\int_a^b |y(x)|^p |y^{(n)}(x)|^q dx \leqq K^*(b-a)^{(n)p}\int_a^b |y^{(n)}(x)|^{p+q}\,dx, \tag{5}$$

where

(6) $$K^* = \alpha q^{q\alpha}[n(1-\alpha)/(n-\alpha)]^{p(1-\alpha)}(n!)^{-p}, \qquad \alpha = (p+q)^{-1}.$$

PROOF. By Hölder's inequality with indices $p+q$ and its conjugate, from

$$|y(x)| \leqq \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}|y^{(n)}(t)|\,dt$$

follows

$$|y(x)| \leqq A(x-a)^{n-\alpha}\left(\int_a^x |y^{(n)}(t)|^{p+q}dt\right)^{\alpha},$$

where A is a constant.

Thus,

$$\int_a^b |y(x)|^p|y^{(n)}(x)|^q dx \leqq B\int_a^b (x-a)^{p(n-\alpha)}|y^{(n)}(x)|^q \cdot\left(\int_a^x |y^{(n)}(t)|^{p+q}dt\right)^{p/(p+q)} dx.$$

Applying Hölder's inequality with indices $(p+q)/p$ and $(p+q)/q$ to the integral on the right-hand side, one has

$$\int_a^b |y^{(k)}(x)|^p|y^{(n)}(x)|^q dx \leqq K^*(q\alpha)^{-q\alpha}\left(\int_a^b |y^{(n)}(x)|^{p+q}\left(\int_a^x |y^{(n)}(t)|^{p+q}dt\right)^{p/q} dx\right)^{q/(p+q)},$$

which is indeed (5).

REMARK 1. For $n=1$, (5) is a sharper result similar to Yang's.

REMARK 2. In case $p>0$, $q>0$ and $p+q=1$, one has

$$|y(x)| \leqq \frac{1}{(n-1)!}(x-a)^{n-1}\int_a^x |y^{(n)}(t)|\,dt,$$

whence

$$\int_a^b |y(x)|^p|y^{(n)}(x)|^q dx \leqq \frac{1}{((n-1)!)^p}\int_a^b (x-a)^{p(n-1)}|y^{(n)}(x)|^q\left(\int_a^x |y^{(n)}(t)|\,dt\right)^p dx.$$

Applying Hölder's inequality with indices $1/p$ and $1/q$ and simplifying

$$(7) \qquad \int_a^b |y(x)|^p |y^{(n)}(x)|^q dx \leqq \frac{q^q}{(n!)^p}(b-a)^{np}\int_a^b |y^{(n)}(x)|\, dx,$$

it is easy to check that K^* in (6) tends to $q^q(n!)^{-p}$ as $p+q\to 1$ from above. Therefore, (5) can be extended to include this case. In (7), *equality holds if and only if* $y^{(n)}(x)\equiv 0$ *in* (a, b) *or* $n=1$ *and* $y^{(n-1)}(x) = \text{const}(x-a)^q$. Indeed, if $y^{(n)}(x)\not\equiv 0$ in (a, b),

$$\int_a^x (x-t)^{n-1}|y^{(n)}(t)|\, dt < \int_a^x (x-a)^{n-1}|y^{(n)}(t)|\, dt, \quad x > a, n > 1.$$

Thus case $n=1$ need only be considered. Equality holds in (7) only if

$$|y^{(n-1)}(x)| = \int_a^x |y^{(n)}(t)|\, dt,$$

and

$$|y^{(n)}(x)| \left(\int_a^x |y^{(n)}(t)|\, dt\right)^{p/q} = \text{const}.$$

The latter leads to

$$\int_a^x |y^{(n)}(t)|\, dt = \text{const}\,(x-a)^q,$$

whence, in view of the former, the assertion follows. (The integrals in (7) are to be interpreted as $\lim_{\epsilon\to 0+}\int_{a+\epsilon}^b$.)

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