STURMIAN THEOREMS FOR HYPERBOLIC EQUATIONS

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Sturm's Comparison Theorem allows two simple physical interpretations, one of which leads to Sturmian Theorems for elliptic equations and the other, as we shall see below, to Sturmian Theorems for hyperbolic equations. Consider first the equations

$$-u'' = pu \tag{1a}$$

$$-v'' = qv \tag{2a}$$

where $u(x_1)=u(x_2)=0$ and $u(x)>0$, $q(x)\geqq p(x)>0$ for $x_1<x<x_2$. Equation (1a) represents a string of density $p(x)$ tied down at $x=x_1$ and $x=x_2$ and vibrating with unit frequency in its fundamental mode. Equation (2a) represents a heavier string, which need only be elastically bound at $x=x_1$ and $x=x_2$, also vibrating with unit frequency. The fact that the second string cannot be vibrating in its fundamental mode is one physical interpretation of Sturm's Theorem. Generalizing this physical argument to membranes instead of strings, one is led in a natural way to Sturmian Theorems for elliptic equations. For a fairly complete bibliography on this subject, see [1], [2].

In order to motivate a generalization to hyperbolic equations, consider a particle of unit mass which is attracted to the origin by a time dependent force proportional to the distance from the origin. The motion of such a particle is described by

$$\ddot{u} + p(t)u = 0. \tag{1b}$$

Given a second particle of unit mass which is attracted to the origin by a greater force of the same type, its motion is described by

$$\ddot{v} = q(t)v = 0, \tag{2b}$$

where $q(t)\geqq p(t)$.

It is physically plausible that between every two successive passes through the origin by the first particle, the second particle will pass through the origin at least once. The mathematical proof of this fact is precisely Sturm's Comparison Theorem: if $q(t)\geqq p(t)$, then the zeros of $v(t)$ separate the zeros of $u(t)$.

This physical interpretation can be used to motivate a generalization of Sturm's Theorem to hyperbolic partial differential equations as follows. Consider a string of unit density and unit elasticity which

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is attracted to the line $u=0$ by a force proportional to the string's displacement from that line. The motion of such a string is described by

$$u_{tt} - u_{xx} + p(x, t)u = 0. \tag{3}$$

Given a second string of unit density and elasticity and attracted to the same line by a greater force of the same type, its motion is described by

$$v_{tt} - v_{xx} + q(x, t)v = 0, \tag{4}$$

where $q(x, t) \geqq p(x, t)$. By analogy to the case of particles, one would expect that the second vibrating string should oscillate faster than the first and that this fact should be the consequence of a Sturmian Theorem for hyperbolic equations.

We shall consider (3) and (4) for $x_1 \leqq x \leqq x_2$ and $t \geqq t_0$ and assume that p and q are continuous in this strip. Our first observation is that a comparison theorem for (3) and (4) will require some auxiliary conditions relating the initial conditions satisfied by solutions of (3) and (4).

LEMMA. *Suppose q does not depend on x. Given any $M>0$ we can choose $f(x)$ so that the solution v of the problem*

$$\begin{gathered} v_{tt} - v_{xx} + q(t)v = 0, \\ v(x, t_0) = 0, \qquad v_t(x, t_0) = f(x) \end{gathered} \tag{5}$$

has no zero for $t_0 < t < M$.

PROOF. If q does not depend on x, we can solve (4) by separation of variables. Writing $v(x, t) = X(x)T(t)$, (4) yields

$$X'' = \lambda X, \qquad T'' + qT = \lambda T$$

and the initial conditions of (5) imply that

$$T(t_0) = 0, \qquad X(x)T'(t_0) = f(x).$$

Given M, we choose a positive constant $q_0 \geqq \sup_{t_0 \leq t \leq M} q(t)$, and let $f(x)$ be a positive solution of $X'' = q_0 X$. Then the solution of (5) is

$$v(x, t) = f(x)T_1(t),$$

where $T_1(t)$ is the solution of

$$T'' + qT = q_0 T, \qquad T(t_0) = 0, \qquad T'(t_0) = 1.$$

Since $q_0 \geqq q(t)$, $T_1(t)$ is positive for $t_0 < t \leqq M$ and $v(x, t)$ has no zero for $t_0 < t \leqq M$.

Thus we cannot expect a generalization of Sturm's Separation Theorem unless some constraints are imposed relating the boundary conditions satisfied by $u(x, t)$ and $v(x, t)$. One means of imposing appropriate constraints is to consider strings which are elastically bound at $x=x_1$ and $x=x_2$—i.e., to prove a comparison theorem for solutions of hyperbolic initial-boundary value problems. Our principal result states that if $q(x)\geqq p(x)$ and if the second string is more tightly bound at x_1 and x_2, then the second string will oscillate more rapidly, in a sense to be made precise below.

THEOREM 1. *Let $u(x)$ be a solution of*

$$u_{tt} - u_{xx} + pu = 0, \qquad u(x, t_0) = 0$$

which is positive for $t_0<t<T$ and satisfies

$$\begin{aligned} u(x, T) &= 0; \qquad x_1 \leqq x \leqq x_2, \\ u_x(x_1, t) - \sigma_1(t)u(x_1, t) &= 0, \\ u_x(x_2, t) + \sigma_2(t)u(x_2, t) &= 0. \end{aligned}$$

If

$$\begin{gathered} q(x, t) \geqq p(x, t) \\ \tau_i(t) \geqq \sigma_i(t); \qquad t_0 \leqq t \leqq T; \quad i = 1, 2, \end{gathered} \tag{6}$$

then every solution of

$$\begin{aligned} v_{tt} - v_{xx} + qv &= 0, \\ v_x(x_1, t) - \tau_1(t)v(x_1, t) &= 0, \\ v_x(x_2, t) + \tau_2(t)v(x_2, t) &= 0 \end{aligned} \tag{7}$$

has a zero in

$$\overline{D} = \{(x, t) \mid x_1 \leqq x \leqq x_2;\ t_0 \leqq t \leqq T\}.$$

PROOF. Suppose to the contrary that $v(x, t)>0$ in $\overline{D}$. Multiplying through (3) and (4) by v and u respectively, and subtracting, we get

$$(vu_t - uv_t)_t - (vu_x - uv_x)_x = (q - p)uv.$$

Integrating this relation over $\overline{D}$ and applying Green's Theorem yields

$$\int_{\partial D} (vu_t - uv_t)dx + (vu_x - uv_x)dt = -\iint_D (q - p)uvdxdt.$$

As a result of the boundary conditions satisfied by u and v, the boundary integral becomes

$$\int_{(x_1,t_0)}^{(x_2,t_0)} vu_t dx + \int_{(x_2,t_0)}^{(x_2,T)} (\tau_2 - \sigma_2) uvdt \tag{8}$$
$$- \int_{(x_1,T)}^{(x_2,T)} vu_t dx - \int_{(x_1,t_0)}^{(x_1,T)} (\sigma_1 - \tau_1) uvdt.$$

From (8) it follows that our hypotheses assure that the boundary integral is positive, contradicting the condition

$$-\iint_D (q - p) uvdxdt \leqq 0.$$

REMARKS. 1. If equality is ruled out in (6), then it is easy to show that v has a zero in the interior of $\overline{D}$.

2. If $v(x, t)$ satisfies $v(x_1, t) = v(x_2, t) = 0$ (corresponding to $\tau_1 = \tau_2 = +\infty$) then no boundary conditions need to be considered for $u(x, t)$.

Consider now the case where (3) allows a separation of variables and the time dependent component is oscillatory at $t = +\infty$. Applying Theorem 1, one can derive oscillation criteria for a class of hyperbolic equations. This idea is the basis of the following theorem which is similar to the results of [3] for the elliptic case.

THEOREM 2. *Let $u(x, t)$ satisfy*

$$\begin{aligned} u_{tt} - u_{xx} + pu &= 0, \\ u_x(x_1, t) - \sigma_1 u(x_1, t) &= 0, \\ u_x(x_2, t) + \sigma_2 u(x_2, t) &= 0, \end{aligned} \tag{9}$$

where p is a function of t only and σ_1, σ_2 are constants. Let λ_1 be the first eigenvalue of

$$-d^2y/dx^2 = \lambda y, \quad y'(x_1) - \sigma_1 y(x_1) = 0, \quad y'(x_2) + \sigma_2 y(x_2) = 0, \tag{10}$$

and suppose that $T'' + (p + \lambda_1)T$ is oscillatory at $t = \infty$. If $v(x, t)$ satisfies (8) *with*

$$q(x, t) \geqq p(t); \qquad \tau_i(t) \geqq \sigma_i, \quad i = 1, 2,$$

then $v(x, t)$ has a zero in

$$F_M = \{(x, t) \mid x_1 \leqq x \leqq x_2;\ t \geqq M\}$$

for every $M > 0$.

PROOF. Setting $u(x, t) = X(x)T(t)$, we can solve (9) by separation of variables to get

$$X'' = \lambda X, \qquad T'' + pT = \lambda T.$$

Choosing X to be the eigenfunction of (10) corresponding to λ_1, one gets

$$T'' + (p + \lambda_1)T = 0$$

which is oscillatory by hypothesis. Letting $\{t_j\}$ denote the zeros of

$$T'' + (p + \lambda_1)T = 0, \qquad T(t_0) = 0$$

one obtains a sequence of closed domains

$$\overline{D}_k = \{(x, t) \mid x_1 \leqq x \leqq x_2;\ t_k \leqq t \leqq t_{k+1}\}$$

in each of which one can apply Theorem 1 to conclude that $v(x, t)$ has a zero in $\overline{D}_k$.

REMARKS. 1. If p is a function of x only, the technique of Theorem 2 also applies.

2. The above results can be extended readily to any pair of self-adjoint ultra-hyperbolic equations with a common principal part.

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