REFLEXIVITY OF CYCLIC BANACH SPACES

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In a Banach space $\mathfrak{X}$ with an unconditional basis $\{e_n\}$ the projections $E(\sigma)$; $\sigma \subset N = \{1, 2, 3, \cdots, n, \cdots\}$ defined by $E(\sigma)(\sum_{n=1}^{\infty} \alpha_n e_n) = \sum_{n\in\sigma} \alpha_n e_n$; $\sum_{n=1}^{\infty} \alpha_n e_n \in \mathfrak{X}$ form a σ-complete *atomic* Boolean algebra of projections $\mathcal{E}$ for which there exists a vector $x_0 \in \mathfrak{X}$ (for instance, $x_0 = \sum_{n=1}^{\infty} e_n/2^n \|e_n\|$) such that $\mathfrak{X} = \operatorname{clm}\{Ex_0 \mid E \in \mathcal{E}\}$. Viewed from this point, the Banach spaces having unconditional basis form a subclass of the family of *cyclic spaces* $\mathfrak{X} = \operatorname{clm}\{Px_0 \mid P \in \mathcal{B}\}$ for some $x_0 \in \mathfrak{X}$ and a σ-complete (not-necessarily atomic) Boolean algebra of projections $\mathcal{B}$ on $\mathfrak{X}$. Cyclic spaces have been introduced by W. G. Bade [1], [2] in connection with the multiplicity theory for spectral operators on Banach spaces. A typical example is $L_1(0, 1)$, the space of all integrable functions on $[0, 1]$, which has no unconditional basis (cf. A. Pełczyński [13, Proposition 9]) but is a cyclic space with respect to the Boolean algebra of projections consisting of "multiplications" by characteristic functions.

W. G. Bade suggested recently in a discussion that it might follow from the theory of normed lattices that a cyclic space is reflexive provided its second conjugate is separable. Using a theorem of T. Ogasawara [12] on normed Riesz spaces we shall be able to prove in the present note that reflexivity of a cyclic space $\mathfrak{X}$ is insured by the condition (weaker than separability of the second conjugate) that neither l_1 nor c_0 would be isomorphic to a subspace of $\mathfrak{X}$. This result generalizes a well-known characterization of reflexivity for spaces with unconditional bases given by R. C. James [5].

Other properties of Banach spaces in connection with Boolean algebras of projections have been described recently in [6], [7], [10], [14].

1. **Preliminaries.** In this section we shall summarize briefly some notion and results needed in the sequel. A Boolean algebra of projections $\mathcal{B}$ is called complete (cf. W. G. Bade [1]) if for every family $P_\alpha \in \mathcal{B}$ the projections $\vee P_\alpha$ and $\wedge P_\alpha$ exist in $\mathcal{B}$ and satisfy

$$(\vee P_\alpha)\mathfrak{X} = \operatorname{clm}\{P_\alpha \mathfrak{X}\}; \qquad (\wedge P_\alpha)\mathfrak{X} = \bigcap (P_\alpha \mathfrak{X}).$$

If $\mathcal{B}$ is complete then there is a uniform bound M for the norms of the projections $P \in \mathcal{B}$ (cf. W. G. Bade [1, Theorem 2.2]). Regarding $\mathcal{B}$ as

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a spectral measure $P(\cdot)$ on the Borel sets Σ of its Stone space Ω, it follows from N. Dunford [3] that for every bounded Borel function f, the integral $S(f)=\int_\Omega f(\omega)P(d\omega)$ exists in the uniform operator topology and satisfies:

$$\|S(f)\| \leqq 4M \sup_{\omega\in\Omega} |f(\omega)| .$$

If f is not bounded and $e_m=\{\omega \mid \omega\in\Omega,\ |f(\omega)| \leqq m\}$, $m=1, 2, \cdots$, the operator $S(f)$ is unbounded having the domain

$$D(S(f)) = \left\{x \,\middle|\, x\in\mathfrak{X},\ \lim_{m\to\infty}\int_{e_m} f(\omega)P(d\omega)x \text{ exists}\right\}.$$

In presenting definitions and results concerning normed lattices we will make use of the terminology and references from W. A. J. Luxemburg and A. C. Zaanen [8, Notes VI and XIII]. Accordingly, a real Banach space L is called a complete Riesz normed space if it is partially ordered by $\leqq$ such that:

(i) $u\leqq v$ implies $u+w\leqq v+w$ for every $u, v, w\in L$.

(ii) $u\geqq 0$ implies $\alpha u\geqq 0$ for every $\alpha\geqq 0$.

(iii) For every pair $u, v\in L$, the least upper bound $\sup(u, v)$ and the greatest lower bound $\inf(f, g)$ exist in L.

(iv) The norm satisfies $\|u\|\leqq\|v\|$ if $|u|\leqq|v|$ (where $|u|=\sup(u,-u)$).

A Riesz space L is said to be σ-Dedekind complete if every sequence in L which is bounded from above has a least upper bound. The notations $u_\tau\downarrow 0$ for a net $\{u_\tau\}\subset L$ means $\{u_\tau\}$ is a decreasing net whose greatest lower bound is zero.

The following theorem due to T. Ogasawara [12, Chapter V, §4, Theorem 1] is stated here in the form found in W. A. J. Luxemburg and A. C. Zaanen [8, Note XIII, Theorem 40.1].

THEOREM A. *A complete Riesz normed space L is reflexive if and only if the following three conditions are satisfied*:

(a) $u_\tau\downarrow 0$ *implies* $\|u_\tau\|\to 0$ *for every net* $\{u_\tau\}\subset L$.

(b) $\phi_\tau\downarrow 0$ *implies* $\|\phi_\tau\|\to 0$ *for every net* $\{\phi_\tau\}\subset L^*$ (L^* *is the conjugate of* L).

(c) $u_\tau\geqq 0$ *and* $\sup\|u_\tau\|<+\infty$ *implies* $\sup_\tau u_\tau\in L$ *for every increasing net* $\{u_\tau\}\subset L$.

2. **Reflexivity of $\mathfrak{M}(x_0)$.** Throughout this section $\mathfrak{B}$ will denote a complete Boolean algebra of projections on the Banach space $\mathfrak{X}$ for which there exists $x_0\in\mathfrak{X}$ such that

$$\mathfrak{X} = \mathfrak{M}(x_0) = \operatorname{clm}\{Px_0 \mid P\in\mathfrak{B}\}.$$

The uniform bound for the norm of the projections $P \in \mathfrak{B}$ will be denoted by M. According to W. G. Bade [2, Theorem 4.5],

$$\mathfrak{X} = \{S(f)x_0 \mid x_0 \in D(S(f))\}.$$

LEMMA 1. *For each* $x \in \mathfrak{X}$, *define* $|x|$ *by* $|x| = \sup \|S(\phi)x\|$ *where the supremum is taken over all Borel functions* ϕ *for which* $|\phi(\omega)| \leqq 1$; $\omega \in \Omega$.

Then $|\cdot|$ *is a norm on* $\mathfrak{X}$ *equivalent to the original norm* $\|\cdot\|$ *and such that*:

(a) $\|x\| \leqq |x| \leqq 4M\|x\|$; $x \in \mathfrak{X}$.

(b) *If* $S(f_1)x_0 \in \mathfrak{X}$ *and* $f_1(\omega) \geqq f_2(\omega) \geqq 0$; $\omega \in \Omega$ *for some Borel function* f_2 *then* $S(f_2)x_0 \in \mathfrak{X}$ *and* $|S(f_1)x_0| \geqq |S(f_2)x_0|$.

(c) $|P| \leqq 1$; $P \in \mathfrak{B}$.

(d) $|S(f)x_0| = |S(|f|)x_0|$; $S(f)x_0 \in \mathfrak{X}$.

The proof follows immediately from the definition of $|x|$ and properties of operators $S(f)$, and we omit it.

Denote $\mathfrak{X}^{(r)} = \{x \in \mathfrak{X} \mid x = S(f)x_0 \mid f \text{ real}\}$. Obviously $\mathfrak{X}^{(r)}$ is a real Banach space which can be ordered by setting $S(f_1)x_0 \preceq S(f_2)x_0$ whenever $f_1(\omega) \leqq f_2(\omega)$ a.e. in Ω. Let us remark that $\mathfrak{X}^{(r)}$ can be considered as the "real part" of $\mathfrak{X}$. In view of Lemma 1, part (b) and the Lebesgue Dominated Convergence Theorem for vector measures (cf. [4, IV-10-10]) we have the following lemma.

LEMMA 2. $\{\mathfrak{X}^{(r)}, \preceq\}$ *is a complete Riesz normed space. Moreover, it is σ-Dedekind complete.*

Now, denote as usual by c_0 the space of sequences convergent to zero and by l_1 the space of sequences whose series are absolutely convergent.

LEMMA 3. *If no subspace of* $\mathfrak{X}^{(r)}$ *is isomorphic to* c_0 *then for every increasing sequence* $0 \preceq S(f_1)x_0 \preceq S(f_2)x_0 \preceq \cdots$ *with* $\sup_n |S(f_n)x_0| < +\infty$ *we have* $x_0 \in D(S(\sup_n f_n))$, *i.e.* $S(\sup_n f_n)x_0 \in \mathfrak{X}^{(r)}$.

PROOF. Assume there exists in $\mathfrak{X}^{(r)}$ an increasing sequence $0 \preceq S(g_1)x_0 \preceq S(g_2)x_0 \preceq \cdots$ with $|S(g_n)x_0| \leqq K$, $n = 1, 2, \cdots$ such that $g(\omega) = \sup_n g_n(\omega)$ is not integrable with respect to the vector measure $P(\cdot)x_0$. According to W. G. Bade [2, Theorem 4.3], there exists a functional $x_0^* \in (\mathfrak{X}^{(r)})^*$ such that $\mu(\cdot) = x_0^* P(\cdot)x_0$ is a positive measure equivalent to the vector measure $P(\cdot)x_0$. Since

$$\int_\Omega g_n(\omega)\mu(d\omega) \leqq K\|x_0^*\|, \qquad u = 1, 2, \cdots,$$

by Fatou's Lemma (cf. [4, III-6-19]), g is integrable with respect to

μ and therefore it is finite a.e. in Ω. Consequently $\Omega=\bigcup_{m=1}^{\infty}\delta_m$ where $\delta_m=\{\omega|\omega\in\Omega, m-1\leqq g(\omega)<m\}$. By a theorem of Lusin (cf. [4, III-6-3]) we can assume with no loss of generality that $\{g_n\}$ converges μ-uniformly to g, i.e., there exists a sequence of disjoint Borel sets $\{\sigma_p\}$ such that $\Omega=\bigcup_{p=1}^{\infty}\sigma_p$ and $\{g_n\}$ converges uniformly to g on every set σ_p, $p=1, 2, \cdots$. The subsets $\delta_m\cap\sigma_p$, $m, p=1, 2, \cdots$ form a sequence of disjoint subsets of Ω which will be denoted by $\{\eta_k\}$. Obviously, $\Omega=\bigcup_{k=1}^{\infty}\eta_k$.

If $\eta_k=\delta_{m_k}\cap\sigma_{p_k}$ let us set

$$\phi_j=\sum_{k=1}^{j}m_k\chi_{\eta_k};\qquad j=1, 2, \cdots,$$

where χ_η denotes the characteristic function corresponding to the set η. It follows immediately that $x_0\in D(S(\phi_j))$, $j=1, 2, \cdots$; $0\preceq S(\phi_1)x_0 \preceq S(\phi_2)x_0\preceq\cdots$ and

$$|S(\phi_j)x_0| = \left|\sum_{k=1}^{j}m_kP(\eta_k)x_0\right| \leqq \left|\sum_{k=1}^{j}(m_k-1)P(\eta_k)x_0\right| + \left|P\left(\bigcup_{k=1}^{j}\eta_k\right)x_0\right|.$$

Hence, by Lemma 1 we have

$$|S(\phi_j)x_0| \leqq \left|\sum_{k=1}^{j}\int_{\eta_k}g(\omega)P(d\omega)x_0\right| + |x_0|$$

and since the convergence is uniform on $\bigcup_{k=1}^{j}\eta_k$

$$|S(\phi_j)x_0| \leqq K+|x_0|,\qquad j=1, 2, \cdots.$$

Furthermore, $\phi(\omega)=\sup_j\phi_j(\omega)=m_k\geqq g(\omega)$ for $\omega\in\eta_k$ which implies in view of Lemma 1, part (b) that $x_0\notin D(S(\phi))$. Thus, the sequence $\{S(\phi_j)x_0\}$ has no limit. The following arguments are similar to those used by R. C. James in [5, Lemma 1]. Since the sequence $\{S(\phi_j)x_0\}$ is not convergent one can easily construct an increasing sequence of integers $\{j_n\}$ such that $|S(\phi_{j_{n+1}})x_0-S(\phi_{j_n})x_0|\geqq\epsilon$; $n=1, 2, \cdots$ for some $\epsilon>0$. Set $\psi_n=\phi_{j_{n+1}}-\phi_{j_n}$ and remark that the functions ψ_n have disjoint supports and

$$\epsilon\leqq|S(\psi_n)x_0|\leqq 2(K+|x_0|),\qquad n=1, 2, \cdots.$$

For any sequence $(\alpha_n)\in c_0$ observe that

$$\sum_{n=p}^{q}\alpha_n\psi_n(\omega)\leqq\max_{p\leqq n\leqq q}|\alpha_n|\,\phi_{j_{q+1}}(\omega),$$

which implies in view of Lemma 1, part (b) that

$$\left|\sum_{n=p}^{q} \alpha_n S(\psi_n)x_0\right| \leqq \max_{p\leq n\leq q} |\alpha_n| (K + |x_0|).$$

Consequently, $\sum_{n=1}^{\infty}\alpha_n S(\psi_n)x_0$ converges and

$$\left|\sum_{n=1}^{\infty} \alpha_n S(\psi_n)x_0\right| \leqq (K + |x_0|) \sup_n |\alpha_n| .$$

On the other hand, according to Lemma 1, part (c)

$$\left|\sum_{n=1}^{\infty} \alpha_n S(\psi_n)x_0\right| \geqq |\alpha_n| |S(\psi_n)x_0| \geqq \epsilon |\alpha_n| , \qquad n = 1, 2, \cdots ,$$

i.e.

$$\left|\sum_{n=1}^{\infty} \alpha_n S(\psi_n)x_0\right| \geqq \epsilon \sup_n |\alpha_n| .$$

Thus the subspace $\operatorname{clm}\{S(\psi_n)x_0;\ n=1, 2, \cdots\}$ is isomorphic to c_0, which contradicts our hypothesis. Q.E.D.

The next step will be to study $\mathfrak{X}^*$, the conjugate of $\mathfrak{X}$. Let $S(f)^* = \int f(\omega)P^*(d\omega)$ be the adjoint of the closed, densely defined operator $S(f)=\int f(\omega)P(d\omega)$, $D(S(f)^*)$ its domain and $x_0^*\in(\mathfrak{X}^{(r)})^*$ the functional already introduced in the proof of the previous lemma whose construction is given by W. G. Bade [2, Theorem 4.3]. According to W. G. Bade [2, Theorem 8.4]

$$\mathfrak{X}^* = \{S(f)^* x_0^* \mid x_0^* \in D(S(f)^*)\}$$

and hence $(\mathfrak{X}^{(r)})^* = \{S(f)^*x_0^* \mid x_0^*\in D(S(f)^*),\ f \text{ real}\}$. One can easily see that $(\mathfrak{X}^{(r)})^*$ with the order $S(f_1)^*x_0^*\preceq S(f_2)^*x_0^*$ whenever $f_1(\omega) \leqq f_2(\omega)$ a.e. in Ω is also a complete Riesz normed space (see also W. A. J. Luxemburg and A. C. Zaanen [8, Note VII, Theorem 22.5]).

Lemma 4. *If no subspace of $\mathfrak{X}^{(r)}$ is isomorphic to l_1 then for every decreasing sequence $S(f_1)^*x_0^*\succeq S^*(f_2)x_0^*\succeq \cdots$ whose greatest lower bound is 0 we have $\lim_{n\to\infty}|S^*(f_n)x_0^*| =0$.*

Proof. Suppose there exists a decreasing sequence $S(h_n)^*x_0^* \in(\mathfrak{X}^{(r)})^*$ such that $\lim_{n\to\infty}h_n(\omega)=0$ a.e. in Ω and $|S(h_n)^*x_0^*| \geqq\epsilon$ for some $\epsilon>0$. By arguments already used in the proof of the previous lemma we can construct a sequence of Borel sets $\Omega\supset\Omega_1\supset\Omega_2\supset \cdots \supset\Omega_p\supset \cdots$ such that $\{h_n(\omega)\}$ converges uniformly for $\omega\in\Omega_p{}'= \Omega-\Omega_p$, $p=1, 2, \cdots$, and $\bigcap_{p=1}^{\infty} \Omega_p=\emptyset$. Obviously for every p there exists an integer n_p (and we can assume that $n_1<n_2<\cdots<n_p<\cdots$) for which $|S(h_{n_p}\chi_{\Omega'_p}{}'')^*x_0^*|<\epsilon/2$. Thus

$$|S(h_1\chi_{\Omega_p})^*x_0^*| \geqq |S(h_{n_p}\chi_{\Omega_p})^*x_0^*| \geqq |S(h_{n_p})^*x_0^*| - |S(h_{n_p}\chi_{\Omega'_p}{}'')^*x_0^*|$$
$$\geqq \epsilon - \epsilon/2 = \epsilon/2, \qquad p = 1, 2, \cdots.$$

Therefore we can find vectors $x_p=S(g_p)x_0\in\mathfrak{X}^{(r)}$ with $|S(g_p)x_0|=1$ and such that

$$[S(h_1)^*x_0^*][S(g_p\chi_{\Omega_p})x_0] \geqq \epsilon/4, \qquad p = 1, 2, \cdots.$$

Consequently $[S(h_1)^*x_0^*][S(|g_p|\chi_{\Omega_p})x_0]\geqq\epsilon/4, p=1, 2, \cdots$.

Since in general the functions $|g_p|\chi_{\Omega_p}$ have no disjoint supports one can find an increasing sequence of integers $\{p_s\}$ such that the functions $\phi_s=|g_{p_s}|\chi_{\Omega_{p_s}-\Omega_{p_{s+1}}}$; $s=1, 2, \cdots$, have disjoint supports and

$$[S(h_1)^*x_0^*][S(\phi_s)x_0] \geqq \epsilon/8; \qquad s = 1, 2, \cdots.$$

Hence, for any sequence $(\alpha_s)\in l_1$ we obtain

$$\begin{aligned} |S(h_1)^*x_0^*| \sum_{s=1}^{\infty} |\alpha_s| &\geqq |S(h_1)^*x_0^*| \left| \sum_{s=1}^{\infty} \alpha_s S(\phi_s)x_0 \right| \\ &\geqq |S(h_1)^*x_0^*| \left| S\left(\sum_{s=1}^{\infty} \alpha_s\phi_s \right) x_0 \right| \\ &= |S(h_1)^*x_0^*| \left| S\left(\sum_{s=1}^{\infty} |\alpha_s| \phi_s \right) x_0 \right| \\ &\geqq [S(h_1)^*x_0^*]\left[S\left(\sum_{s=1}^{\infty} |\alpha_s| \phi_s \right) x_0 \right] \\ &= \sum_{s=1}^{\infty} |\alpha_s| [S(h_1)^*x_0^*][S(\phi_s)x_0] \geqq \frac{\epsilon}{8} \sum_{s=1}^{\infty} |\alpha_s|, \end{aligned}$$

i.e., l_1 is isomorphic to the subspace $\operatorname{clm}\{S(\phi_s)x_0;\ s=1, 2, \cdots\}$, which is a contradiction. Q.E.D.

THEOREM 5. *The cyclic space $\mathfrak{X}=\mathfrak{M}(x_0)$ is reflexive if and only if no subspace of it is isomorphic to either l_1 or c_0.*

PROOF. Since every subspace of a reflexive space is also reflexive no subspace of $\mathfrak{X}$ can be isomorphic to l_1 or c_0 provided $\mathfrak{X}$ is reflexive. To prove the converse notice first that it suffices to show that $\mathfrak{X}^{(r)}$ is reflexive. For this purpose we shall use Theorem A. Indeed, condition (a) of this theorem holds in view of a theorem of H. Nakano [**11**, pp.

321–322] (see also W. A. J. Luxemburg and A. C. Zaanen [8, Note X, Theorem 33.4]), the Lebesgue Dominated Convergence Theorem for vector measures and the fact that $\mathfrak{X}^{(r)}$ is σ-Dedekind complete. Condition (b) follows from W. A. J. Luxemburg and A. C. Zaanen [8, Note X, Theorem 33.8], (used for $(\mathfrak{X}^{(r)})^*$), again the Lebesgue Dominated Convergence Theorem, and Lemma 4 provided no subspace of $\mathfrak{X}^{(r)}$ is isomorphic to l_1. Finally, if no subspace of $\mathfrak{X}^{(r)}$ is isomorphic to c_0, Lemma 3 and W. A. J. Luxemburg and A. C. Zaanen [8, Note XI, Theorem 34.2] imply that condition (c) is also satisfied. This completes the proof.

COROLLARY 6. *If $\mathfrak{X}^{**}$ is separable then $\mathfrak{X}$ is reflexive.*

PROOF. If $\mathfrak{X}$ has a subspace isomorphic to either c_0 or l_1 then $\mathfrak{X}^{**}$ cannot be separable since $(c_0)^{**}=m$ and $l_1^*=m$ (m denotes the space of all bounded sequences) and m is not separable.

REMARKS. 1. This corollary can be proved directly by using Lemma 2 and another result of T. Ogasawara [12, Chapter V; §4, Theorem 3] (see also W. A. J. Luxemburg [9, Theorem 45.1]).

2. Corollary 6 is not true for an arbitrary Banach space (cf. R. C. James [5]).

3. In connection with the proof of Lemma 4, one can observe that if e_s denotes the support of ϕ_s then

$$PS(f)x_0 = \sum_{s=1}^{\infty} \frac{[S(h_1)^* x_0^*][S(f\chi_{e_s})x_0]}{[S(h_1)^*x_0^*][S(\phi_1)x_0]} S(\phi_s)x_0$$

is a bounded projection (with norm$\leq (8/\epsilon)|S(h_1)^*x_0^*|$) onto the subspace clm$\{S(\phi_s)x_0; s=1, 2, \cdots\}$ which is isomorphic to l_1. Similar arguments show that in Lemma 3 the subspace clm$\{S(\phi_n)x_0;\ n=1, 2, \cdots\}$ (which is isomorphic to c_0) is also the range of a projection with norm$\leq (K+|x_0|)/\epsilon$.

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