used under (i) implies $z \leqq 0$ on S_T. Hence $M \leqq 0$. Similarly by considering $-w$, the minimum is nonnegative. Thus $w = 0$.

Bibliography

1. A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall Englewood Cliffs, N. J., 1964.

2. M. Protter and H. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, N. J., 1967.

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AN ELEMENTARY DERIVATION OF KHINTCHINE'S ESTIMATE FOR LARGE DEVIATIONS

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1. **Introduction.** In classical proofs of the law of the iterated logarithm, the estimate

$$P(S_n/\sqrt{n} \geqq a_n) = \exp[-(a_n^2/2)(1 + o(1))] \qquad (n \uparrow \infty) \tag{1.1}$$

plays a key role (see [3, pp. 41–49]). Here S_n is a sum of n independent identically distributed random variables with mean zero and variance one; $\{a_n\}$ is a fixed numerical sequence with some growth property. The first direct proof [2] of inequalities of this type involved cumbersome estimates of bilateral Laplace transforms and was restricted to bounded random variables. More recently, proofs of (1.1) and related inequalities have been derived as a corollary to global inequalities of the Berry-Essen type:

$$P\left(\frac{S_n}{\sqrt{n}} \geqq a\right) = \int_a^\infty \frac{\exp(-t^2/2)}{(2\pi)^{1/2}}\, dt + O(n^{-1/2}) \qquad (n \uparrow \infty) \tag{1.2}$$

when the error is uniform in $a \in (-\infty, \infty)$. The key observation in these proofs is that for a suitable choice of $a = a_n$, the error term in (1.2) can be absorbed into the Gaussian term (see [1, pp. 212–219], and [4]).

The purpose of this note is to point out that the idea of absorbing the error can be applied to a (much more easily proved) smoothed version of (1.2) to yield (1.1). The proof is based on Trotter's method of operators [5], which is presented in the lemma below. The whole point is that while Trotter's method seems incapable of yielding

Received by the editors July 15, 1968.

[1] Research supported in part under contract N0014-67-A-0112-0015 at Stanford University, Stanford, California.

(1.2), the estimation of "not too large" deviations is insensitive to the approximation of a unit step function by a smooth function.

2. **Proof of the inequalities (1.1).** Let $\{X_n\}_{n\geq 1}$ be independent random variables with the same distribution; we make the normalizations $E(X_n)=0$, $E(X_n^2)=1$, and assume that for some $\delta>0$, $E(|X_n|^{2+\delta})<\infty$; let $S_n=X_1+\cdots+X_n$.

LEMMA. *If f has three bounded continuous derivatives, we have*

$$\left|E\left[f\left(\frac{S_n}{\sqrt{n}}\right)\right]-\int_{-\infty}^{\infty}f(x)\frac{\exp(-x^2/2)}{(2\pi)^{1/2}}dx\right| \leqq K\frac{\|f\|}{n^{\delta/2}}E(|X_1|^{2+\delta})$$

where K depends only on δ and $\|f\|=\sup_x[|f''(x)|+|f'''(x)|]$.

PROOF. Let $\{g_n\}_{n\geq 1}$ be independent gaussian random variables with mean zero, variance one, completely independent of $\{X_n\}_{n\geq 1}$. Then

$$\begin{aligned} E[f(S_n/\sqrt{n})] &- E[f(g_1)] \\ &= E[f((X_1+\cdots+X_n)/\sqrt{n})] - E[f((g_1+\cdots+g_n)/\sqrt{n})] \\ &= \sum_{i=1}^{n}\left\{E\left[f\left(\frac{B_{i,n}+X_i}{\sqrt{n}}\right)\right]-E\left[f\left(\frac{B_{i,n}+g_i}{\sqrt{n}}\right)\right]\right\} \\ &= \sum_{i=1}^{n}A_i \end{aligned}$$

where $B_{i,n}=g_1+\cdots+g_{i-1}+X_{i+1}+\cdots+X_n$. If we bring in the modified Taylor estimate:[2] $|f(x+y)-f(x)-yf'(x)-(y^2/2)f''(x)| \leqq |y|^{2+\delta}\|f\|$, it follows that $|A_i|\leqq\|f\|\times n^{-1-\delta/2}\times E[|X_i|^{2+\delta}+|g_i|^{2+\delta}]$ which is of the required form.

THEOREM. *If $\{a_n\}$ is a sequence increasing to $+\infty$ so that $a_n^2-\log n\to-\infty$, then for each $\epsilon>0$*

$$\exp\{-(a_n^2/2)(1+\epsilon)\}\leqq P(S_n/\sqrt{n}\geqq a_n)\leqq\exp\{(-a_n^2/2)(1-\epsilon)\} \tag{2.1}$$

for $n\geqq N(\epsilon)$.

PROOF. Let $f_n^{\pm}(x)=f_0(x-a_n\mp 1/2)$ where f_0 is a fixed C_3 function vanishing for $x\leqq -1/2$, equal to one for $x\geqq 1/2$ with $0\leqq f_0\leqq 1$. Let p_n be the middle term in (2.1); $\Phi(x)$ denotes the tail integral $(2\pi)^{-1/2}\int_x^{\infty}\exp(-u^2/2)du$. Then clearly

[2] To prove this, consider separately the cases $|y|>1$ and $|y|\leqq 1$ and apply the two and three term Taylor expansions respectively.

$$E[f_n^-(S_n/\sqrt{n})] \leqq p_n \leqq E[f_n^+(S_n/\sqrt{n})]. \tag{2.2}$$

If we apply the lemma to the extreme members of (2.2) and then over (respectively under) estimate $f_n^\pm$ by an indicator function, it follows that

$$\Phi(a_n + 1) - \overline{K}n^{-\delta/2} \leqq p_n \leqq \Phi(a_n + 1) + \overline{K}n^{-\delta/2} \tag{2.3}$$

where $\overline{K}$ is independent of n. If we now use the well-known estimate for the tail: $\log \Phi(a_n \pm 1) = -a_n^2/2\ (1+o(1))$, it becomes clear that the hypothesis on $\{a_n\}$ is equivalent to $n^{-\delta/2}/\Phi_n \to 0$, where $\Phi_n = \Phi(a_n \pm 1)$. Thus we have $\log p_n/\log \Phi(a_n) \to 1$ and hence the result.

3. **Extensions of the method.** Let $\{X_n\}_{n\geq 1}$ be independent random variables with mean zero and variance σ_n^2; let $S_n = X_1 + \cdots + X_n$, $s_n^2 = \sigma_1^2 + \cdots + \sigma_n^2$, $r_n = s_n^{-(2+\delta)} \sum_{k=1}^n E[|X_k|^{2+\delta}]$. The above method easily generalizes to show that if for some $\delta > 0$, $r_n \to 0$ then we have $P(S_n/s_n \geqq a_n) = \exp[-a_n^2/2\ (1+o(1))]$ for any numerical sequence $\{a_n\}$ for which $a_n^2/\log(1/r_n) \to 0$ as $n \uparrow \infty$.

References

1. K. L. Chung, *A course in probability theory*, Harcourt Brace, New York, 1968.
2. A. Y. Khintchine, *Asymptotische Gesetze der Wahrscheinlichkeitrechung*, Ergebnisse der Math., Vol. 2, Springer-Berlin, Berlin, 1933.
3. J. Lamperti, *Probability*, Benjamin, New York, 1966.
4. V. V. Petrov, *On a relation between an estimate of the remainder in the central limit theorem and the law of the iterated logarithm*, Theor. Probability Appl. **11** (1966), 454–458.
5. H. F. Trotter, *An elementary proof of the central limit theorem*, Arch. Math. **9** (1959), 226–234.

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