COMMUTATIVE RINGS WHOSE MATRIX RINGS ARE BAER RINGS

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A ring R with unit element is a Baer ring if every left annihilator in R has the form Re, where e is an idempotent element. K. G. Wolfson has proven [**3**, Corollary 15], that if R is a Prüfer ring (a commutative integral domain in which every finitely generated ideal is invertible) then the ring of endomorphisms of a finitely generated free module over R is a Baer ring. In this note we view these endomorphism rings as the matrix rings R_n with which they are isomorphic, and show that under the rather modest assumption that R have descending chain condition on annihilators, the converse of this result holds. It is also shown that in this case all matrix rings over R are Baer rings if any one of them is.

THEOREM. *Let R be a commutative ring with unit element and descending chain condition on annihilators. Then the following are equivalent*:

(1) *R_n is a Baer ring for every $n \geqq 2$.*

(2) *R_k is a Baer ring for some particular $k \geqq 2$.*

(3) *R is a finite direct sum of Prüfer rings.*

PROOF. To show that (3) implies (1), let $R = R_1 \oplus \cdots \oplus R_t$, where the R_i are Prüfer rings. Then since the matrix ring $R_n = (R_1)_n \oplus \cdots \oplus (R_t)_n$ and finite direct sums of Baer rings are Baer rings, this reduces to Wolfson's result. That (1) implies (2) is obvious.

To establish that (2) implies (3), suppose that R_k is a Baer ring for some specific integer $k \geqq 2$. Define a matrix $(e_{ij}) \in R_k$ by $e_{11} = e_{22} = 1$, $e_{ij} = 0$ otherwise. This matrix is idempotent, and therefore $(e_{ij})R_k(e_{ij})$ is a Baer ring [**2**, Theorem 2, p. 3]. Since this subring of R_k is isomorphic to R_2, we may assume from the outset that $k = 2$.

We wish to show first that R is a direct sum of integral domains, and so it may be assumed that R contains zero-divisors. Let $A \neq (0)$ be a minimal annihilator in R. This ideal exists because R is not a domain. Since the center of a Baer ring is again a Baer ring [**2**, Corollary to Theorem 3, p. 4], $A = Re$, where $e^2 = e \neq 0$. Suppose that $a, b \in A$, $b \neq 0$ and $ab = 0$. Then $\operatorname{ann}(a) \cap A \neq (0)$, so by the minimality of A, $A \subseteq \operatorname{ann}(a)$, whence $aA = (0)$. Putting $a = re$ for an appropriate $r \in R$, we have $Rre = reRe = aA = (0)$, so $a = re = 0$ and A is an integral domain. Let $A_1 = A$, $B_1 = R(1-e)$; clearly $R = A_1 \oplus B_1$. The ring B_1 also has descending chain condition on annihilators. If B_1 has divisors

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of zero, then since the matrix ring $R_2=(A_1)_2\oplus(B_1)_2$ and a direct summand of a Baer ring is a Baer ring [2, Corollary to Theorem 2, p. 3], we may repeat the process on B_1 to get $B_1=A_2\oplus B_2$ where A_2 is an integral domain. Then $R=A_1\oplus A_2\oplus B_2$. Continuing in this way, one obtains a descending chain of annihilators $\{B_i\}$, and thus by hypothesis there must exist an integer t such that $B_t=B_{t+1}=\cdots$. This can only mean that B_t is an integral domain, since otherwise the procedure could be continued. Hence $R=A_1\oplus\cdots\oplus A_t\oplus B_t$ is a direct sum of integral domains.

Since it is clear from the facts quoted above that each of these summands has a 2-by-2 matrix ring which is a Baer ring, we shall have completed the proof of the theorem if we show that whenever R is a commutative integral domain such that R_2 is a Baer ring, then R is a Prüfer ring. Let R be such a domain.

Suppose that P is any prime ideal of R, and that R_P is the localization of R at P. Let $T\in(R_P)_2$ be a zero-divisor in the 2-by-2 matrix ring over R_P. Write the entries of T over a common denominator a, so that $T=(a_{ij}/a)$, a_{ij}, $a\in R$, $a\neq 0$. It is clear that the matrix (a_{ij}) is a zero-divisor of R_2, and hence by hypothesis there is an idempotent $(e_{ij})\neq 0$ such that $(e_{ij})(a_{ij})=0$. Then in $(R_P)_2$, $(e_{ij})T=0$, so $(R_P)_2$ has the property that the left annihilator of any zero-divisor contains a nonzero idempotent. Let a and b be any two nonzero elements of R_P. Form the matrix $(a_{ij})\in(R_P)_2$ which has entries $a_{11}=a$, $a_{21}=b$, $a_{12}=a_{22}=0$; then (a_{ij}) is a zero-divisor, so there exists an idempotent $(e_{ij})\neq 0$ with $(e_{ij})(a_{ij})\neq 0$. Notice that not all of the e_{ij} can be in P_P, the maximal ideal of R_P, for the Jacobson radical of $(R_P)_2$ is exactly the set of matrices all of whose entries come from P_P, and the radical of a ring cannot contain nonzero idempotents. The equality $(e_{ij})(a_{ij})=0$ yields the two equations $e_{11}a+e_{12}b=0$, $e_{21}a+e_{22}b=0$. But one of the e_{ij} is not in P_P and therefore is a unit of R_P. Multiplication of the appropriate one of these two equations by the inverse of this e_{ij} displays one of a or b as a multiple of the other.

Hence for any prime ideal P, R_P has the property that whenever a and b are in R_P and are not zero, then one of them is a multiple of the other, i.e. R_P is a valuation ring. It is well known [1, Exercise 12, p. 93] that an integral domain with this property has every finitely generated ideal invertible, and therefore we have shown that R is a Prüfer ring.

Observing that a commutative noetherian ring has descending chain condition on annihilators and that a noetherian Prüfer ring is a Dedekind domain, we obtain an interesting special case of the theorem.

COROLLARY. *Let R be a commutative noetherian ring. Then every R_n is a Baer ring if and only if R is a direct sum of Dedekind domains.*

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