ON FINITELY GENERATED SUBGROUPS OF A FREE GROUP

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1. **Introduction.** In [2], M. Hall, Jr. proved the following theorem: Let H be a finitely generated subgroup of a free group F and suppose $\beta_1, \cdots, \beta_n$ are in F but no β_i is in H. Then we may construct a subgroup $\overline{H}$ of finite index in F containing H and not containing any β_i.

Now the proof in [2] actually shows more, viz., that H is a free factor of $\overline{H}$. In particular, taking the set of β_i's to be empty one obtains the following:

If H is a finitely generated subgroup of a free group F, then H is a free factor of a subgroup $\overline{H}$ of finite index in F.

In this paper we shall show how a number of results about finitely generated subgroups of a free group follow in a natural way from the above special case of the theorem of M. Hall, Jr. In particular, we derive the following: a finitely generated subgroup H is of finite index in F iff H has a nontrivial intersection with every nontrivial normal subgroup of F (this includes the case, see [4], where H contains a nontrivial normal subgroup of F, and the case, see L. Greenberg [1], when H contains a nontrivial subnormal subgroup of F); a generalization of this for a pair of subgroups H, K (see Theorem 3, Corollary 1); other types of conditions for a finitely generated H to be of finite index in F (first proved by L. Greenberg in [1] for discrete groups of motions of the hyperbolic plane (which include free groups); and Howson's result that the intersection of two finitely generated subgroups of F is finitely generated. We also derive a quick way of obtaining the precise index of H in F from inspection of a Nielsen reduced set of generators for H.

2. **The results.** In what follows "f.g." denotes "finitely generated" and "f.i." denotes "finite index."

If H, $\overline{H}$ are subgroups of a group G, then H is a *factor of* $\overline{H}$ if $\overline{H}$ is the free product $H * H_1$ for some $H_1 < G$; $\overline{H}$ is a *completion of H in G* if H is a factor of $\overline{H}$ and $\overline{H}$ is of f.i. in G. By the above result of M. Hall, Jr., every f.g. subgroup H of a free group F has a completion $\overline{H}$ in F.

LEMMA 1. *Let H be a factor of $\overline{H}$ in G and let $K < G$. Then $H \cap K$ is a factor of $\overline{H} \cap K$.*

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Proof. Since $\overline{H}$ is a free product, say $\overline{H}=H * H_1$, we may apply the Kurosh subgroup theorem to the subgroup $\overline{H}\cap K$ and conclude that $\overline{H}\cap K$ is a free product of factors one of which is its intersection with H, viz., $H\cap(\overline{H}\cap K)=H\cap K$. [Remark: If G is a free group, then Lemma 1 can be proved without using the Kurosh subgroup theorem; see, for example, [6, p. 117 exercise 32].]

Corollary. *If $\overline{H}$, $\overline{K}$ are completions of H, K respectively in G, then $\overline{H}\cap\overline{K}$ is a completion of $H\cap K$ in G.*

Proof. Clearly $\overline{H}\cap\overline{K}$ is of f.i. in G. Moreover, by Lemma 1, $H\cap K$ is a factor of $\overline{H}\cap K$ which itself is a factor of $\overline{H}\cap\overline{K}$.

Theorem 1 (Howson). *If H, K are f.g. subgroups of a free group, then $H\cap K$ is also f.g.*

Proof. Let $\overline{H}$ be a completion of H in F. Then $\overline{H}\cap K$ is of f.i. in K and so is f.g. Moreover, since $H\cap K$ is a factor of $\overline{H}\cap K$ (by Lemma 1), we have that $H\cap K$ is a homomorphic image of $\overline{H}\cap K$ and so $H\cap K$ is f.g.

Lemma 2. *If G is the free product $A * B$ $(B\neq 1)$ and N is a subnormal subgroup of G contained in A, then $N=1$.*

Proof. Suppose $L\neq 1$ is a subgroup of A. Then the normalizer of L in G is contained in A, for $v^{-1}av$ is in A implies v is in A (as is easily seen by using the reduced form of v in the free product). Now suppose

$$N_r \lhd N_{r-1} \lhd \cdots \lhd N_1 \lhd G$$

and $N_r\neq 1$, $N_r<A$. Then N_{i-1} is contained in the normalizer of N_i and therefore $G<A$, contrary to $B\neq 1$.

Theorem 2. (Greenberg). *If H is a f.g. subgroup of a free group F and H contains a subnormal subgroup $N\neq 1$ of F, then H is of f.i. in F.*

Proof. Let $\overline{H}$ be a completion of H in F. Then $N<\overline{H}$ and N is a subnormal subgroup of $\overline{H}=H * H_1$. Hence by Lemma 2, $H_1=1$ and so $H=\overline{H}$, i.e. H is of f.i. in F.

Lemma 3. *If $G=A * B$ $(A\neq 1, B\neq 1)$, then A has trivial intersection with some noncyclic normal subgroup of G, viz., the normal subgroup of G generated by B.*

Proof. Let $a\in A$, $b\in B$ with $a\neq 1$, $b\neq 1$, and let N be the normal subgroup of G generated by B. Then under the natural homomorphism of $G\rightarrow G/N$, A is mapped isomorphically onto itself and so $A\cap N=1$. Moreover, since b and aba^{-1} are in N and do not commute, N is noncyclic.

THEOREM 3. *Let H be a f.g. subgroup $\neq 1$ of a free group F. Then the following conditions are equivalent:*

(a) *H is of f.i. in F.*

(b) *H has a nontrivial intersection with every nontrivial normal subgroup of F.*

(c) *H is not properly contained in a subgroup of rank larger than that of H.*

PROOF. Clearly (a) implies that H has nontrivial intersection with every infinite subgroup and so (a) implies (b); (a) implies (c) follows from the Schreier rank formula, since $j(n-1)+1 \geqq n$ (see, for example, [6, p. 104]).

Conversely, suppose (c). Hence if $\overline{H}=H * H_1$ is a completion of H in F, then $H_1=1$, and $H=\overline{H}$ is of f.i. in F.

Finally, suppose (b). Again let $\overline{H}=H * H_1$ be a completion of H in F and suppose $H_1 \neq 1$. Let N be the normal subgroup of $\overline{H}$ generated by H_1. By Lemma 3, $N \cap H=1$. Since the normalizer of N in F contains $\overline{H}$, N has finitely many conjugates in F. The intersection of these conjugates is therefore nontrivial (see, for example, [5, p. 219]), is normal in F, and is disjoint from H, contrary to hypothesis. Consequently, $H_1=1$ and $H=\overline{H}$ is of f.i. in F.

[Theorem 3 is a generalization of Theorem 2 because *any two nontrivial subnormal subgroups of a free product intersect nontrivially.* For, let N, K be nontrivial normal subgroups of G. Suppose $N \cap K=1$; then every element of N commutes with every element of K. If N contains an element $\neq 1$ in A, then $K<A$ (see, for example, [6, Corollary 4.1.6]), contrary to Lemma 2. Hence we may assume N and K have trivial intersection with the conjugates of A and B. Then the elements of N and K are of infinite order and $\exists n \in N$, $k \in K$, $n \neq 1$, $k \neq 1$, such that $nk=kn$ and so n, k are both powers of the same element. But then $N \cap K \neq 1$.

Next suppose

$$1 \neq N_r \lhd N_{r-1} \lhd \cdots \lhd N_1 \lhd G$$

and

$$1 = K_s \lhd K_{s-1} \lhd \cdots \lhd K_1 \lhd G.$$

We show by induction on $r+s$, the total length of the two normal chains, that $N_r \cap K_s \neq 1$. If $r+s=2$, then N_r and K_s are normal in G. Suppose $r+s>2$, and say $s \geqq 2$. Since N_r and K_1 occur in normal chains of total length $r+1<r+s$, by inductive hypothesis $N_r \cap K_1 \neq 1$. Moreover, K_1 is either infinite cyclic or a proper free product (by Lemma 2 and the Kurosh subgroup theorem). Hence, since in K_1,

$N_r \cap K_1$ and K_s occur in normal chains of total length $r+s-1$, we have $(N_r \cap K_1) \cap K_s = N_r \cap K_s \neq 1$.]

COROLLARY 1. *Let H, K be subgroups of a free group F with H f.g. Then the following conditions are equivalent:*

(a) *$H \cap K$ is of f.i. in K.*

(b) *H contains a nontrivial subnormal subgroup N of K.*

(c) *H has a nontrivial intersection with every nontrivial normal subgroup of K.*

PROOF. Clearly (a) implies (b) and (c).

Suppose (b). Let $\overline{H}$ be a completion of H in F. Then $H \cap K$ contains N and N is subnormal in $\overline{H} \cap K$. By Lemma 1, $H \cap K$ is a factor of $\overline{H} \cap K$ and so by Lemma 2, $\overline{H} \cap K = H \cap K$. Clearly $\overline{H} \cap K$ is of f.i. in K.

Suppose (c) and again let $\overline{H}$ be a completion of H in F. Then $\overline{H} \cap K = (H \cap K) * L_1$ and arguing as in the proof of the above theorem we obtain that $H \cap K = \overline{H} \cap K$, which is of f.i. in K.

COROLLARY 2 (GREENBERG). *Let $H \neq 1$ be a f.g. subgroup of a free group F. Then the following conditions are equivalent:*

(a) *H is of f.i. in F.*

(b) *H has a nontrivial intersection with every noncyclic subgroup of F.*

(c) *H is not contained in a subgroup of infinite rank.*

PROOF. Clearly (a) implies (b) and (c).

Conversely, suppose (b). Since every nontrivial normal subgroup of a noncyclic free group is noncyclic, H satisfies (b) of the above theorem and hence (a) holds.

Finally, suppose (c). Let $\overline{H} = H * H_1$ be a completion of H in F. If $H_1 \neq 1$ then N, the normal subgroup of $\overline{H}$ generated by H, has infinite index in $\overline{H}$ and therefore is infinitely generated.

The terminology in the following theorem is used in the same way as in §3.2 of [6].

THEOREM 4. *Suppose F is the free group on $x_1, x_2, \cdots, x_n$ and H is a f.g. subgroup of F with a Nielsen reduced set of generators $\{W_i(x_\nu)\}$, $1 \leqq i \leqq r$, where the right halves of the W_i of even length are isolated. Let*

$$W_i = U_i x_{\nu_i}^{\epsilon_i} V_i^{-1}$$

where V_i^{-1} is the minor terminal segment of W_i, and let j be the number of distinct initial segments of $U_1, V_1, \cdots, U_r, V_r$. Then H is of f.i. in F iff $j = (r-1)/(n-1)$ iff $[F: H] = j$.

PROOF. The argument given by M. Hall, Jr. in [2] to prove the result quoted in the introduction can be modified to construct a Schreier coset representative function whose representatives are precisely the distinct initial segments of $U_1, V_1, \cdots, U_r, V_r$ and such that the representative of $U_i x_{\nu_i}^{\epsilon_i}$ is V_i. The associated subgroup $\overline{H}$ clearly has index j in F and has H as a factor. Now H is of f.i. in F iff $H=\overline{H}$ iff the rank of H equals the rank of $\overline{H}$. But the rank of $\overline{H}$ is $j(n-1)+1$. Therefore H is of f.i. in F iff $r=j(n-1)+1$, i.e., $j=(r-1)/(n-1)$.

Note that a given Nielsen reduced set of generators $\{W_i(x_\nu)\}$ may be converted into one with isolated right halves simply by replacing W_j by W_j^{-1} if necessary. The above test requires that the right halves be isolated as the following example shows: Let F be the free group on a, b and let $H=gp(a^2, ab, ab^{-1})=gp(a^2, ab, ba^{-1})$. Then both sets of generators for H are Nielsen reduced. Now in this case $n=2$, $r=3$ and so by the above test H is of f.i. iff $j=2$. The first set of generators has isolated right halves and $U_1=a$, $V_1=1$, $U_2=a$, $V_2=1$, $U_3=a$, $V_3=1$, so that $j=2$; hence H is of f.i. 2 (indeed, H is just the subgroup consisting of words of even length in F). On the other hand, the second set of generators does not have isolated right halves and an attempt to apply the test in this case yields $U_1=a$, $V_1=1$, $U_2=a$, $V_2=1$, $U_3=b$, $V_3=1$, which leads to $j=3$ and the incorrect conclusion that H is not of f.i.

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