

THE CONNECTION BETWEEN TWO GEOMETRICAL AXIOMS OF H. N. GUPTA

WOLFRAM SCHWABHÄUSER

The answer is given here to a problem raised by H. N. Gupta at the end of [1], namely, if two axioms (E) and (B) are equivalent when added to Euclidean geometry of arbitrary dimension over arbitrary ordered fields. (B) states that, for any points x, y, z such that y is between x and z , there is a right triangle having x and z as endpoints of the hypotenuse and y as foot of the perpendicular drawn from the right angle vertex. (B) holds in the n -dimensional Cartesian space $\mathbb{E}_n(\mathfrak{F})$ over an ordered field \mathfrak{F} ($n \geq 2$) if and only if

(b) for all $a_1, \dots, a_n, l \in \mathfrak{F}$ with $l \geq 0$, the system of the two equations

$$(1) \quad a_1 x_1 + \dots + a_n x_n = 0,$$

$$(2) \quad x_1^2 + \dots + x_n^2 = l \cdot (a_1^2 + \dots + a_n^2)$$

has a solution with x_1, \dots, x_n in \mathfrak{F} .

(E) is, essentially, a combination of the axiom of segment construction and the circle axiom. It is analytically expressed (as above) by (e) \mathfrak{F} is Euclidean (i.e., in \mathfrak{F} , each positive element is a square).

As stated in [1], (E) implies (B) (also for infinite dimension), and the converse holds in Euclidean geometry over Pythagorean ordered fields.

It will be shown here that (B) does not imply (E) in general, in fact, that we get a counterexample, satisfying (b), by taking the field \mathbb{Q} of rationals for \mathfrak{F} together with any $n \geq 5$, while, clearly, (e) does not hold for $\mathfrak{F} = \mathbb{Q}$. Thus, the answer to the above problem is negative.

First, let the ordered field \mathfrak{F} and $n \geq 2$ still be arbitrary. It suffices to consider, in (b), the case that not all a_i are zero. Putting $c = l \cdot (a_1^2 + \dots + a_n^2)$, we get that (b) is equivalent with

(b') for all $a_1, \dots, a_n, c \in \mathfrak{F}$ with $c \geq 0$ and at least one $a_i \neq 0$, the system of the two equations (1) and

$$(2') \quad x_1^2 + \dots + x_n^2 = c$$

has a solution with x_1, \dots, x_n in \mathfrak{F} .

For fixed a_1, \dots, a_n of this kind, the solutions \mathfrak{x} of (1) (in usual vector notation) form an $(n-1)$ -dimensional subspace of \mathfrak{F}^n . Let

Received by the editors August 28, 1968.

$m = n - 1$ and b_1, \dots, b_m be a basis of this subspace. (By applying well-known procedures, we may assume these vectors to be pairwise orthogonal, but not necessarily of the same length since square roots in \mathfrak{F} need not exist.) Thus, (1) and (2') (which can be written as $\mathfrak{x}^2 = c$) have a common solution in \mathfrak{F}^n if and only if

$$(3) \quad \left(\sum_{\mu=1}^m t_\mu b_\mu \right)^2 = c$$

has a solution $\langle t_1, \dots, t_m \rangle$ in \mathfrak{F}^m . (3) is equivalent with

$$(4) \quad \sum_{\lambda, \mu=1}^m g_{\lambda\mu} t_\lambda t_\mu = c \quad \text{where } g_{\lambda\mu} = b_\lambda b_\mu.$$

The left-hand side of (4) (or (3)) is a positive definite quadratic form in t_1, \dots, t_m .

Hence, (b) certainly holds if

(b⁺) *for each positive definite quadratic form in $m = n - 1$ variables over \mathfrak{F} , each positive element c of \mathfrak{F} is in the range of this form (i.e., a solution of (4) in \mathfrak{F}^m exists).*

Now, by Hasse [2, Satz 22, p. 147], (b⁺) holds for $\mathfrak{F} = \mathfrak{Q}$ and $m \geq 4$, i.e. $n \geq 5$. Thus, also (b) holds in this case, as was to be proved.

REMARK. Putting $a_1 = 1, a_2, \dots, a_n = 0$, we get that (b) implies

(b⁻) *in \mathfrak{F} , each nonnegative element c is a sum of $n - 1$ squares.*

By Hasse [2, Beispiel 1, p. 145], not each positive rational number can be represented as a sum of three squares of rationals, hence, (b) does not hold for $\mathfrak{F} = \mathfrak{Q}$ and $n = 4$.

Thus, for $\mathfrak{F} = \mathfrak{Q}$, 5 is the least possible value for n in our counter-example; in other words, 5 is the least dimension n such that Gupta's axiom (B) holds in $\mathfrak{C}_n(\mathfrak{Q})$.

REFERENCES

1. H. N. Gupta, *On some axioms in foundations of Cartesian spaces*, Canad. Math. Bull. (to appear).
2. H. Hasse, *Über die Darstellbarkeit von Zahlen durch quadratische Formen im Körper der rationalen Zahlen*, J. Reine Angew. Math. 152 (1923), 129–148.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT BONN, GERMANY