AN ASYMPTOTIC ANALOG OF THE F. AND M. RIESZ RADIAL UNIQUENESS THEOREM[1]

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Let $D=\{|z|<1\}$, and let $C=\{|z|=1\}$. We say that $f(z)$, defined in D, has the radial limit a at $e^{i\theta}\in C$ if $\lim_{r\to 1} f(re^{i\theta})=a$. One such theorem concerning radial limits for bounded holomorphic functions is the classical F. and M. Riesz uniqueness theorem:

THEOREM. *If $f(z)$ is holomorphic and bounded in D and has the radial limit zero at each point of a subset of C of positive measure, then $f\equiv 0$.*

The Riesz theorem is not true for sets of measure zero since it is known that given any set $P\subset C$ of measure zero (in particular, P may be of second category) there exists a nonconstant function $f(z)$, holomorphic and bounded in D, with radial limit zero at each point of P [**4**, p. 214]. However, the following theorem shows that something very definitely can be said about the rate at which a bounded holomorphic function approaches zero along radii terminating at a subset of C of second category.

THEOREM 1. *Let $\mu(r)$ be any positive monotone decreasing function on $[0, 1)$ such that $\lim_{r\to 1}\mu(r)=0$. Let S be any subset of C of second category. If $f(z)$ is any function bounded and holomorphic in D with the property $|f(re^{i\theta})|=o(\mu(r))$ for each $e^{i\theta}\in S$, then $f\equiv 0$.*

PROOF. First note that by the main theorem in [**1**, p. 6] there exists a function $g(z)$, holomorphic and nonconstant in D, such that

$$\max_{0\leq\theta<2\pi} |g(re^{i\theta})| < 1/(\mu(r)) \quad\text{and}\quad \lim_{r\to 1} g(re^{i\theta}) = 0$$

for $e^{i\theta}\in T$, where T is a subset of C of measure 2π. Consider the function $h(z)=f(z)\cdot g(z)$. Now note that $\lim_{r\to 1} h(re^{i\theta})=0$ for $e^{i\theta}\in S\cup T$. Since $S\cup T$ is of measure 2π and second category, $h(z)$ satisfies the hypotheses of the Lusin and Priwalow radial uniqueness theorem [**4**, p. 232], and we see that $h(z)\equiv 0$. Hence $f(z)\equiv 0$.

REMARK 1. Note the hypothesis of the theorem does *not* say that $f(z)$ goes uniformly to zero along radii terminating at points of S (in that case the theorem is trivially true for arbitrary holomorphic

Presented to the Society, April 20, 1968; received by the editors October 18, 1968.

[1] This research was partially supported by the National Science Foundation, Grant GP-8787.

functions) but only that along each such radius $|f(re^{i\theta})|$ *eventually* goes to zero faster than $\mu(r)$.

REMARK 2. With a little care, minor alterations in the above proof will yield an analogous theorem for meromorphic functions of bounded characteristic. There is also, of course, an analogous theorem for negative harmonic functions which go to minus infinity faster than any given fixed rate on a set of second category.

REMARK 3. Recalling Beurling's uniqueness theorem for univalent functions [**2**, pp. 11–12], which says that if a univalent function $f(z)$ has radial limit zero on a set of positive outer capacity, then $f(z)\equiv 0$. This might lead one to suspect that the correct hypothesis on S in Theorem 1 should be of positive outer capacity. This is not true, however, because given any closed set N $(\subset C)$ of zero measure by the Rudin-Carleson theorem [**3**, p. 81] there exists a nonconstant function $f(z)$ which is bounded and holomorphic in D, continuous in $D\cup C$, and such that $f(e^{i\theta})=0$ if $e^{i\theta}\in N$.

REMARK 4. We can prove an analog of Theorem 1 for arbitrary holomorphic functions if we assume that $\mu(r)$ tends to zero more rapidly than $\exp[-(1-r)^{-1-\epsilon}]$. This statement is stronger than the remark in [**5**, p. 384] but it easily follows from a stronger form of the "Picard-Schottky theorem in an angle," a fact which the second author did not realize at the time.

It is possible that this analog also holds for all $\mu(r)\rightarrow 0$. However, it is in this "gray area," where the function is unbounded but goes to zero no faster than order one, that a counterexample may exist since in this case the function has "lots of room to move around" but is not required to go to zero fast enough to cause "Picard property" behaviour.

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