HARRISON PRIMES IN A RING WITH FEW ZERO DIVISORS

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1. Introduction. In [4] D. K. Harrison introduces preprimes, primes, and valuation preprimes to pave the way for an algebraic number theory in arbitrary commutative rings. He shows [**4**, §3] that his definitions coincide with the classic definitions of valuation theory when restricted to fields. In this paper the author extends Harrison's results to a large class of rings which includes all Noetherian rings. In addition an analog to the approximation theorem for valuations is developed for this class of rings.

Let R be a commutative ring with unity consisting entirely of zero divisors and units and having finitely-many maximal ideals, for example, the total quotient ring of a Noetherian ring. The adjunction of the word "regular" to a nonempty subset of R will mean that the set contains at least one regular element; i.e., a nonzero divisor. Let X be the set of regular finite primes P of R such that the total quotient ring of $P:P$ is R. Then all these subrings have *few zero divisors* [**3**, p. 203]. The terminology and results of [**3**] and [**4**] will be used freely. For any subring A of R and multiplicatively-closed subset S of A consisting of regular elements, we will assume that the ring of quotients A_S is canonically embedded in R. For two subrings A, B of R, $A:B=\{x\in R \mid xB\subseteq A\}$. In placing the above restriction on R and X, the author is prescribing a situation resembling the situation involved in the classical valuation theory in fields.

Give X the topology generated by taking as basic open sets all sets of the form $\{P\in X \mid P\cap E=\varnothing\}$ where E is a finite set of *regular* elements of R. This is *not* the same topology as Harrison uses in [**4**, p. 10]. Let $X^\#$ be the set of closed irreducible subsets C of X such that $\bigcap C=\bigcap\{P \mid P\in C\}$ is regular. Give $X^\#$ the topology which takes sets of the form $\{C\in X \mid C\cap V\neq\varnothing\}$, for some open subset V of X, as open sets. Let $\mathfrak{X}$ be the set of all regular valuation preprimes S of R such that the total quotient ring of $S:S$ is R and topologize $\mathfrak{X}$ by using $\{S\in\mathfrak{X} \mid S\cap E=\varnothing\}$, for a finite set E of regular elements of R, as basic open sets. A *regular valuation pair* (A, P) of R is simply a valuation pair (A, P) with P regular.

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2. Extension of Harrison's results.

THEOREM 1. *Let P be a regular preprime of R and let $A=P:P$. P is a valuation preprime of R with R as the total quotient ring of A if and only if (A, P) is a valuation pair of R.*

The proof of this assertion is embodied in the following propositions which will show that $X^\#$ and $\mathfrak{X}$ are homeomorphic under the maps $C\in X^\#\to\bigcap C$ and $P\in\mathfrak{X}\to\{Q\in X \mid P\subseteq Q\}$. The continuity of these maps is easy to check after the propositions establish their one-oneness.

REMARKS. (1) The definition of certain valuation pairs can be weakened. Let (A, P) be such that A is a subring of R whose total quotient ring is R, P a regular prime ideal of A, and for all *regular* $x\in R\setminus A$, there exists $y\in P$ such that $xy\in A\setminus P$. Then (A, P) is a valuation pair of R, for if $z\in R\setminus A$, then there exists $w\in P$ such that $z+w$ is regular. Then $z+w\in R\setminus A$ implies the existence of $y\in P$ such that $(z+w)y\in A\setminus P$. Hence $zy\in A\setminus P$. The existence of the $w\in P$ such that $z+w$ is regular follows by an argument similar to that used by Davis [**3**, Lemma B, p. 204] and using the fact that P is a *regular* ideal of A.

(2) A similar result is true for regular valuation preprimes. Let S be a regular preprime such that $-S\subseteq S$, the total quotient ring of $S:S$ is R, and for any finite set E of *regular* elements of R such that $S\cap E=\varnothing$, there exists a finite prime P of R such that $S\subseteq P$ and $P\cap E=\varnothing$. Let $E=\{e_1, \cdots, e_n\}$ be any subset of R. Then there exists $u_1, \cdots, u_n\in S$ such that $\{e_1+u_1, \cdots, e_n+u_n\}$ is a set of regular elements disjoint from S. Hence there exists a finite prime P such that $S\subseteq P$ and $e_i+u_i\notin P$ for $i=1, \cdots, n$. But then $e_i\notin P$ for $i=1, \cdots, n$, so S is a valuation preprime.

(3) The ordering on the set of all pairs (A, P) where A is a subring of R and P a prime ideal of A is $(A_1, P_1)\leqq(A_2, P_2)$ if and only if $A_1\subseteq A_2$ and $P_1=P_2\cap A_1$. Manis [**6**, Definition 1.8] has established that maximal pairs and valuation pairs of R are equivalent terms. For any regular valuation pair (A, P) of R, we have $(A, P)\leqq(A_{S(P)}, PA_{S(P)})$ where $S(P)$ denotes the set of regular elements in the complement of P in A. Hence by the maximality of (A, P), P must be the unique maximal regular ideal of A [**3**, Lemma A, p. 204].

(4) Let (A, P) be a valuation pair of R. For a regular element $x\in R$, if $x, x^{-1}\in R\setminus A$, then there exists $y, z\in P$ such that $xy, x^{-1}z\in A\setminus P$. Thus $yz\in P\cap(A\setminus P)$ which is impossible. Thus x or x^{-1} must be in A. Consequently when (A, P) is a valuation pair of R, A is a *quasi-valuation ring* [**3**, p. 203]. On the other hand, if A is a quasi-valuation ring and P is its unique maximal regular ideal, then Remark 1 shows that (A, P) is a regular valuation pair of R.

(5) Remark 4 and [3, Proposition 5, p. 205] imply that if (A, P) is a valuation pair of R and (B, Q) is a pair where B is a subring of R containing A and Q is the maximal regular ideal of B, then (B, Q) is a valuation pair of R and $B = A_{S(Q)}$.

Proposition 1. *Let $C \in X^{\#}$ with $P = \bigcap C$ and $A = P:P$. Then (A, P) is a valuation pair of R.*

Proof. Let p be a regular element of P. Clearly A is a subring of R and P is an ideal of A. Suppose $a, b \in A$ with $a \notin P$, $b \notin P$, and $ab \in P$. Then there exist $u, v \in A$ such that $c = a + up$, $d = b + vp$ are regular and $c \notin P$, $d \notin P$, and $cd \in P$. Hence $\{Q \in C \mid c \notin Q\}$ and $\{Q \in C \mid d \notin Q\}$ are nonempty open subsets of C whose intersection is empty contrary to the irreducibility of C. Thus P is a prime ideal of A.

Let $x \in R \backslash A$, x regular. Then since $x \notin A$ there exists $y \in P$ such that $xy \notin P$. There exists $u \in A$ such that $y + up$ is regular and $u \in (A:x)$. [This is Davis' argument again using the fact that $A \cap (A:x)$ is a regular ideal of A.] Then $y + up = x^{-1}(x(y+up)) \in P$ and $x(y+up) \notin P$. An argument similar to the one above based on the irreducibility of C shows that the complement of P in R is closed under multiplication of regular elements. Hence $x^{-1} \in P$. Hence for x regular in $R \backslash A$, there exists $x^{-1} \in P$ such that $xx^{-1} \in A \backslash P$. By Remark 1 this proves that (A, P) is a valuation pair of R. Note that $x^{-1} \in A$ for all regular $x \in R \backslash A$ implies that the total quotient ring of A is R.

Proposition 2. *If (A, P) is a valuation pair of R, then P is a valuation preprime of R.*

Proof. Let E be a finite set with $P \cap E = \varnothing$. Let $E = E_1 \cap E_2$ where $E_1 = \{e_1, \cdots, e_s\} \subseteq A$ and $E_2 = \{e_{s+1}, \cdots, e_r\} \subseteq R \backslash A$. For $i = s+1, \cdots, r$, there exist $t_i \in P$ such that $e_i t_i \in A \backslash P$. Let $E' = \{e_1, \cdots, e_s, e_{s+1}t_{s+1}, \cdots, e_r t_r\}$. Then $E' \subseteq A$ so $E'P \subseteq P$. By [4, Lemma 2.6, p. 21], there exists a finite prime Q of R such that $P \subseteq Q$ and $Q \cap E' = \varnothing$. Thus $E_1 \cap Q = \varnothing$ and for $i = s+1, \cdots, r$, $e_i t_i \in Q$ and $t_i \in Q$ imply that $e_i \notin Q$ so $E_2 \cap Q = \varnothing$. Thus $E \cap Q = \varnothing$.

For a valuation preprime P of R, let $C_P = \{Q \in X \mid P \subseteq Q\}$.

Proposition 3. *If P is a valuation preprime of R and $\bigcap C_P$ is regular, then C_P is a closed irreducible subset of X.*

Proof. The irreducibility follows readily. To see that C_P is closed note that C_P is the complement of

$$\bigcup \{Q \in X \mid \{x\} \cap Q = \varnothing \quad \text{for some regular } x \in P\}.$$

One inclusion is clear; for the other, note that for $Q \notin C_P$, there exists

$x\in P$, $x\notin Q$. If p is a regular element of $\bigcap C_P$, then there is a $u\in P:P$ such that $x+up$ is regular, $x+up\in P$, $x+up\notin Q$. Therefore Q is not in the complement mentioned above.

PROPOSITION 4. *If P is a regular valuation preprime of R, then $P=\bigcap C_P$.*

PROOF. Clearly $P\subseteq\bigcap C_P$. Let $x\in R\backslash P$ and let $A=P:P$. If $x\in A\backslash P$, then there exists $Q\in C_P$ such that $x\notin Q$; hence $x\notin\bigcap C_P$. On the other hand, $x\in R\backslash A$. By Proposition 3 C_P is a closed irreducible subset of X. Thus by Proposition 1 (A, P) is a valuation pair of R. Hence there exists $y\in P$ such that $xy\in A\backslash P$. Hence there exists $Q\in C_P$ such that $xy\notin Q$. But $y\in P\subseteq Q$ implies $x\notin Q$. Hence $x\notin\bigcap C_P$.

3. Relationship to integral closure. In this section we show that the integral closure of a ring has the same characterization as in the classical setting. First a lemma.

LEMMA 1. *Let T be a subring of R, I a proper regular ideal of T. Then there exists a valuation pair (A, P) of R such that $I\subseteq P$.*

PROOF. Consider the set of all pairs (A, P) where A is a subring of R and P is a prime ideal of A containing I. This inherits the ordering discussed in Remark 3 and in this ordering has a maximal element which is also maximal in the set of all such pairs with no restriction regarding I. Hence this maximal element is a valuation pair satisfying the assertion of the lemma.

THEOREM 2. *For any subring A of R whose total quotient ring is R, the integral closure $\overline{A}$ of A is*

$$A^* = \bigcap_{\alpha} \{A_\alpha \mid (A_\alpha, P_\alpha) \text{ is a valuation pair of } R,\ A \subseteq A_\alpha\}.$$

PROOF. $\overline{A}\subseteq A^*$ since each A_α is integrally closed [6, Proposition 1.9]. Now let $x\in R\backslash\overline{A}$, x regular. Then x^{-1} is not a unit in $A[x^{-1}]$, for if it were, then $x=a_0+a_1x^{-1}+\cdots+a_sx^{-s}$ for $a_0, a_1, \cdots, a_s\in A$ which would violate $x\notin\overline{A}$. Thus $x^{-1}A[x^{-1}]$ is a proper ideal of $A[x^{-1}]$. Thus, by Lemma 1, there exists a valuation pair (B, P) of R such that $x^{-1}A[x^{-1}]\subseteq P$. Since $x^{-1}\in P$, $x\notin B$. Thus the regular elements of A^* belong to A. Hence by [3, Lemma C, p. 205] $A^*\subseteq\overline{A}$.

4. An approximation theorem for valuations. The following definitions and results are due to Manis [6, Proposition 1.6]. Let (A, P) be a valuation pair of R. For $x, y\in R$, set $x\sim y$ if and only if $P:(x) =P:(y)$. Let $V(x)=\{y\in R\mid y\sim x\}$ and $\Gamma_V=\{V(x)\mid x\in R\}$. Then $\Gamma_V\backslash V(0)$ is a group with identity $V(1)=A\backslash P$ and composition

$V(x)V(y)=V(xy)$ for all $x, y\in R$. Define $V(x)<V(y)$ if there exists $z\in R$ such that $xz\in P$ and $yz\in A\backslash P$. Then $<$ is a linear order on Γ_V, $\Gamma_V\backslash V(0)$ is an ordered group, and $V(x+y)\leqq\max\{V(x), V(y)\}$ for all $x, y\in R$.

We use these facts to prove an approximation theorem for valuations. The approach here imitates that of [2, p. 132–136]. Again, if P is a prime ideal of a ring A, $S(P)$ denotes the set of regular elements in the complement of P in A. Using the previous remarks, [5, Lemma 3.1] may be restated as

THEOREM 3. *Let $\{(A_i, P_i)\}_{i=1,\cdots,n}$ be a set of regular valuation pairs of R. Let $A=\bigcap_{i=1}^n A_i$ and set $Q_i=P_i\cap A$ for $i=1,\cdots,n$. Then $A_i=A_{S(Q_i)}$ for $i=1,\cdots,n$ and if $A_i\not\subseteq A_j$ for $i\neq j$, then $\{Q_i\}_{i=1,\cdots,n}$ is the set of maximal regular ideals of A.*

LEMMA 2. *Let (A, P) be a valuation pair of R, M a proper regular ideal of A, and $N=\sqrt{M}$. Then N is a prime ideal of A.*

PROOF. Let V be the valuation defined by (A, P) and let m be a regular element of M. Let $x, y\in A$ with $xy\in N$. There exist $u, v\in A$ *such that* $s=x+um$, $t=y+vm$ are regular. Then $st\in N$ so $(st)^n\in M$ for some positive integer n. If $V(s)\geqq V(t)$, then $V(ts^{-1})\leqq V(1)$, so $t\in sA$. Hence $t^{2n}\in s^nt^nA\subseteq M$ which proves that $t\in N$.

Following [2, Definition 1, p. 134] we say two valuation pairs (A_1, P_1) and (A_2, P_2) of R are *dependent* if the ring generated by A_1 and A_2 is properly contained in R. Since R has few zero divisors, this is equivalent to the existence of a valuation pair (A, P) of R such that for $i=1, 2$, $A\supseteq A_i\supseteq P_i\supseteq P\supseteq V_i^{-1}(0)$ where V_i is the valuation associated with (A_i, P_i). This second form of this definition is the one used by Manis [6].

THEOREM 4. *Let $\{(A_i, P_i)\}_{i=1,\cdots,n}$ be pairwise independent valuation pairs of R with valuations V_i and groups $\Gamma_i\backslash\{V_i(0)\}$. For $x_1, x_2, \cdots, x_n\in R$ and $\alpha_i\in\Gamma_i\backslash\{V_i(0)\}$ for $i=1,\cdots,n$, there exists $x\in R$ such that $V_i(x-x_i)\leqq\alpha_i$ for $i=1,\cdots,n$.*

PROOF. We may assume that each V_i is proper. Let $A=\bigcap_{i=1}^n A_i$ and $Q_i=A\cap P_i$. Since the total quotient ring of A is R, there exist $a, s, a_1, \cdots, a_n\in A$, s regular, such that $x_i=a_i/s$ and $x=a/s$. Then $V_i(a-a_i)\leqq\alpha_iV_i(s)$ for all i will imply the result. Hence we may assume that $x_1,\cdots,x_n\in A$. Also we may assume that $\alpha_i<V_i(1)$ for all i. Let $M_i=\{z\in R\mid V_i(z)\leqq\alpha_i\}$ and set $N_i=M_i\cap A$. For $x\in A$, $V_i(x-x_i)\leqq\alpha_i$ is equivalent to $x\equiv x_i \pmod{N_i}$. Thus it suffices to show that the canonical map $A\rightarrow\prod_{i=1}^n A/N_i$ is onto; i.e., that the N_i are pairwise

comaximal [1, Proposition 5, p. 72]. By Theorem 3, $Q_1, \cdots, Q_n$ are the maximal regular ideals of A. Thus it suffices to show that $N_i \not\subseteq Q_j$ for $i \neq j$.

Suppose that $N_i \subseteq Q_j$ for some $i \neq j$. Using Lemma 2 one sees that $N = \sqrt{N_i}$ is a prime ideal of A so that $N \subseteq Q_j$. Also since $\alpha_i < V_i(1)$, $N_i \subseteq Q_i$ so that $N \subseteq Q_i$. Hence $A_i = A_{S(Q_i)} \subseteq A_{S(N)}$ and similarly $A_j \subseteq A_{S(N)}$. Each M_i contains a regular element, for let $u \in R$ be such that $V_i(u) = \alpha_i$. Then $V_i(w) = \alpha_i^{-1}$ for some $w \in R$, say $w = x/y$ with $x, y \in A$, y regular. Let p be a regular element of P_i. Then $wu \in A_i \setminus P_i$ and $wyp \in P_i$ showing that $V_i(yp) < \alpha_i$; i.e., that $yp \in M_i$. By [1, Proposition 10, p. 89] $M_i = N_i A_{S(Q_i)}$, so N_i must contain a regular element; therefore N does too and hence $A_{S(N)} \neq R$. This contradicts the independence of A_i and A_j.[2]

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[2] The author has learned that Malcolm Griffin has obtained Remark 4 and Theorem 2 independently in a paper to appear in J. Reine Angew. Math.