SOME TYPES OF BOREL MEASURES

ROY A. JOHNSON

1. **Introduction.** Let X be a locally compact Hausdorff space. Our definitions of Baire sets, Borel sets, Borel measures and regular Borel measures are those of [2]. All measures considered here will be nonnegative Borel measures on X, and μ, ν, and λ are reserved for such measures. (Recall Borel measures in [2] are always σ-finite.) In §2 we show that given a Borel measure μ, we may distill from μ a largest regular measure μ_0 such that $\mu=\mu_0+\mu_1$. The residual measure μ_1 will not only fail to be regular; it will be singular with respect to all regular Borel measures on X. A Borel measure μ will be called monogenic if the only Borel measure agreeing with μ on Baire sets is μ itself. In §3 we show that given a Borel measure μ, we may distill from μ a largest monogenic measure μ_0.

We use the concepts of singularity ($\mu\perp\nu$) and absolute continuity ($\nu\ll\mu$) extensively. The definitions are those of [2, pp. 126 and 124]. The following properties will be used implicitly and explicitly:

I. If $\nu\ll\mu$ and $\lambda\perp\mu$, then $\lambda\perp\nu$.

II. If $\nu\ll\mu$ and $\nu\perp\mu$, then $\nu=0$ [2, Exercise 30.9].

III. If $\mu\leqq\nu+\lambda$ and $\mu\perp\lambda$, then $\mu\leqq\nu$.

IV. If $\mu\perp\lambda$ and $\nu\perp\lambda$, then $(\mu+\nu)\perp\lambda$ [2, Exercise 30.10].

V. If $\mu_n(E)\to\mu(E)$ for each Borel set E and if $u_n\perp\nu$ for each n, then $\mu\perp\nu$.

VI. If (1) $\mu_\alpha(E)\uparrow\mu(E)$ for each Borel set E, (2) $\mu_\alpha\perp\nu$ for each α and (3) μ or ν is regular, then $\mu\perp\nu$ [3, 3.1 and 3.3]. Hence, if (1) $\mu=\sum\mu_\alpha$ or $\mu=\bigvee\mu_\alpha$ (the smallest measure $\geqq$ each μ_α), (2) $\mu_\alpha\perp\nu$ for each α and (3) μ or ν is regular, then $\mu\perp\nu$.

VII. *Lebesgue decomposition theorem* (e.g. [3, 3.4]). Suppose μ and ν are Borel measures on X and that μ or ν is regular. Then there exist unique Borel measures μ_0 and μ_1 such that $\mu=\mu_0+\mu_1$, where $\mu_0\ll\nu$ and $\mu_1\perp\nu$.

Although properties I–V hold for abstract (not necessarily Borel) measures, properties VI and VII do not always hold, even for σ-finite measures. (It is understood that "each Borel set E" is replaced by "each measurable set E" in V and VI.) I do not know if the regularity of μ or ν is essential in VI and VII.

2. **Regular and antiregular measures.** If μ and ν agree on all Baire

Presented to the Society, August 30, 1968; received by the editors October 9, 1968.

sets, we say that μ and ν are *Baire relatives* or simply *relatives.* If μ is a Borel measure, then its restriction to the Baire sets is a Baire measure which can be extended to a unique regular Borel measure μ' [2, 54.D]. Clearly μ' is a relative of μ, which we may call the *regular relative* of μ. Of course $\mu=\mu'$ if and only if μ is regular. Finally, we say that μ is *antiregular* if and only if $\mu\perp\nu$ for each regular measure ν. Evidently the only measure which is both regular and antiregular is the zero measure.

Our first theorem follows immediately from the definition of antiregular measures and the properties of singularity listed earlier.

THEOREM 2.1. *Let $\mathfrak{N}$ be the class of all antiregular Borel measures on X. Then:*

1. *If $\nu\ll\mu$ and $\mu\in\mathfrak{N}$, then $\nu\in\mathfrak{N}$.*
2. *If μ, $\nu\in\mathfrak{N}$, then so are $\mu+\nu$ and $\mu\vee\nu$.*
3. *If $\mu_n(E)\to\mu(E)$ for each Borel set E and if $\mu_n\in\mathfrak{N}$ for each n, then $\mu\in\mathfrak{N}$.*
4. *If $\mu_\alpha\in\mathfrak{N}$ for each α and if the Borel measure μ can be written as $\sum\mu_\alpha$ or $\bigvee\mu_\alpha$ (the smallest measure $\geqq$ each μ_α), then $\mu\in\mathfrak{N}$.*

THEOREM 2.2. *If $\mu\perp\mu'$, where μ' is the regular relative of μ, then μ is antiregular.*

PROOF. Suppose ν is a regular Borel measure on X. Then by the Lebesgue decomposition theorem, we may write $\mu=\mu_0+\mu_1$, where $\mu_0\ll\nu$ and $\mu_1\perp\nu$. Then μ_0 is regular [2, Exercise 52.9]. Now if μ_1' and μ' are the regular relatives of μ_1 and μ, respectively, then it is easy to see that $\mu'=\mu_0+\mu_1'$, so that $\mu_0\leqq\mu'$. But since $\mu\perp\mu'$, it follows that $\mu_0=0$, and we are done.

THEOREM 2.3. *If μ is any Borel measure, then there exist unique regular μ_0 and antiregular μ_1 such that $\mu=\mu_0+\mu_1$.*

PROOF. Let μ' be the regular relative of μ. By the Lebesgue decomposition theorem we have $\mu=\mu_0+\mu_1$, where $\mu_0\ll\mu'$ and $\mu_1\perp\mu'$. Necessarily, μ_0 is regular. We show that μ_1 is antiregular. For, let μ_1' be the regular relative of μ_1. It is clear that $\mu'=\mu_0+\mu_1'$, so that $\mu_1'\leqq\mu'$. Since $\mu_1\perp\mu'$, we have $\mu_1\perp\mu_1'$, which shows that μ_1 is antiregular.

To prove uniqueness, suppose $\mu=\mu_2+\mu_3$, where μ_2 is regular and μ_3 is antiregular. Since $\mu_0\leqq\mu$ and $\mu_0\perp\mu_3$, we have $u_0\leqq\mu_2$ by property III in the Introduction. Similarly, $\mu_2\leqq\mu_0$, so that $\mu_0=\mu_2$ and $\mu_1=\mu_3$.

THEOREM 2.4. *A Borel measure μ is antiregular if and only if there exists a locally Borel set A (i.e. $E\cap A$ is Borel for each Borel set E)*

such that $\mu(E-A)=0$ *for all Borel sets* E *and such that* $\mu(C)=0$ *for each compact set* $C\subset A$.

PROOF. Suppose μ is antiregular. Then $\mu\perp\mu'$, where μ' is the regular relative of μ. Hence there exists a locally Borel A such that $\mu(E-A)=0=\mu'(E\cap A)$ for each Borel set E. Now if C is a compact subset of A, evidently $\mu'(C)=0$. Since there exists a compact G_δ set D such that $\mu'(C)=\mu'(D)$ [**1**, 59.1], we have $0=\mu'(D)=\mu(D)\geqq\mu(C)$, so that $\mu(C)=0$.

On the other hand, suppose there exists such an A as described above. We know that $\mu=\mu_0+\mu_1$, where μ_0 is regular and μ_1 is antiregular, so that it suffices to show that $\mu_0=0$. Suppose to the contrary that $\mu_0(E)>0$ for some Borel set E. Then $\mu_0(E\cap A)>0$ and by regularity of μ_0 there exists a compact $C\subset E\cap A$ such that $\mu_0(C)>0$. Of course $\mu(C)>0$ in this case, and that is impossible since $C\subset A$.

THEOREM 2.5. *If* μ *is antiregular, then* $\mu(\{x\})=0$ *for each* $x\in X$.

PROOF. For each $x\in X$, let $\nu_x(E)=1$ or 0 according to whether $x\in E$ or $x\notin E$. Then ν_x is a regular measure, so that $\mu\perp\nu_x$. It follows that $\mu(\{x\})=0$.

3. **Monogenic and antimonogenic measures.** We shall say that a Borel measure μ is *monogenic* if the only relative of μ is μ itself (cf. [**1**, p. 231]). Such a measure is necessarily regular. We shall say that a Borel measure μ is *antimonogenic* if μ is singular with respect to every monogenic measure. Then Theorem 2.1 holds if $\mathfrak{N}$ is taken to be the class of antimonogenic measures.

Evidently the only measure which is both monogenic and antimonogenic is the zero measure. It is also clear that every antiregular measure is antimonogenic. But there are antimonogenic measures which are not antiregular. Indeed, a necessary and sufficient condition for a Borel measure to be antimonogenic is that it have an antiregular relative. We prove the sufficiency now.

THEOREM 3.1. *If* μ *has an antiregular relative, then* μ *is antimonogenic.*

PROOF. We wish to show that if λ is monogenic, then $\mu\perp\lambda$. We consider first the case in which μ is regular. If D is a Baire set, we define a measure μ_D by $\mu_D(E)=\mu(E\cap D)$ for each Borel set E. By hypothesis, μ has an antiregular relative ν, and it is clear that μ_D and ν_D are relatives since D is a Baire set.

Now suppose D is a fixed Baire set. By the Lebesgue decomposition theorem we may write $\lambda=\lambda_0+\lambda_1$, where $\lambda_0\ll\mu_D$ and $\lambda_1\perp\mu_D$. By the

Radon-Nikodym theorem, there exists a Borel function f such that $\lambda_0(E)=\int_E f\,d\mu_D$ for each Borel set E. Since μ_D is regular, we may assume that f is a Baire function [1, 68.1]. Define $\lambda_2(E)=\int_E f\,d\nu_D$ for each Borel set E. Since $\lambda_2 \ll \nu$, we have λ_2 is antiregular. Since f is a Baire function and since μ_D and ν_D are relatives, it follows that λ_0 and λ_2 are relatives [1, 66.1]. Since $\lambda=\lambda_0+\lambda_1$ and $\lambda_2+\lambda_1$ are relatives and since λ is monogenic, we have $\lambda=\lambda_2+\lambda_1$. But since λ is regular and λ_2 is antiregular, we have $\lambda=\lambda_1$, so that $\lambda \perp \mu_D$.

The measures μ_D are increasingly directed in the obvious sense, and $\mu=\text{LUB}\ \mu_D$. Since $\mu_D \perp \lambda$ for each Baire set D, we have $\mu \perp \lambda$. This completes the case where μ is regular.

Now suppose μ is not necessarily regular. Since μ has an antiregular relative, so does μ', where μ' is the regular relative of μ. Hence μ' is antimonogenic. Now $\mu=\mu_0+\mu_1$, where μ_0 is regular and μ_1 is antiregular. Since $\mu_0 \leqq \mu'$ and $\mu' \perp \lambda$, whenever λ is monogenic, we have $\mu_0 \perp \lambda$. Of course $\mu_1 \perp \lambda$, so that $(\mu_0+\mu_1) \perp \lambda$, as was to be shown.

LEMMA 3.2. *If μ is not monogenic, then there exists a nonzero λ such that $\lambda \leqq \mu$ and such that λ has an antiregular relative.*

PROOF. If μ is not regular, the conclusion is clear. We may assume μ is regular. Then μ has a nonregular relative, ν. We may write $\nu=\nu_0+\nu_1$, where ν_0 is regular and ν_1 is antiregular. If λ is the regular relative of ν_1, then $\nu_0+\lambda=\mu$, so that λ will serve as the required measure.

THEOREM 3.3. *If μ is a Borel measure, then there exist Borel measures μ_0 and μ_1 such that μ_0 is monogenic, μ_1 has an antiregular relative (and hence is antimonogenic) and such that $\mu=\mu_0+\mu_1$. The requirement that μ_0 and μ_1 be monogenic and antimonogenic, respectively, determines them uniquely.*

PROOF. If μ is monogenic, we are done. Otherwise there exists nonzero $\lambda \leqq \mu$ such that λ has an antiregular relative. Now let us say that a family of measures $\{\mu_\alpha\}$ is admissible if

(1) μ_α is nonzero for each α,

(2) μ_α has an antiregular relative ν_α for each α, and

(3) $\sum \mu_\alpha \leqq \mu$.

Let us order the (nonempty) collection of admissible families by inclusion. By Zorn's lemma there exists a maximal admissible family, which we label $\{\mu_\alpha\}$. Let $\mu_1=\sum \mu_\alpha$, and let μ_0 be that unique measure such that $\mu=\mu_0+\mu_1$.

If, for each α, ν_α is an antiregular relative for μ_α, then $\sum \nu_\alpha$ is easily seen to be an antiregular relative of $\mu_1=\sum \mu_\alpha$. Also μ_0 is monogenic in view of Lemma 3.2 and the maximality of the family $\{\mu_\alpha\}$. The uniqueness argument is the same as that in Theorem 2.3, and we omit it.

THEOREM 3.4 (CONVERSE OF THEOREM 3.1). *If μ is antimonogenic, then μ has an antiregular relative.*

PROOF. Suppose μ is antimonogenic. We have $\mu=\mu_0+\mu_1$, where μ_0 is monogenic and μ_1 has an antiregular relative. Since $\mu\perp\mu_0$, we have $\mu=\mu_1$.

THEOREM 3.5. *A Borel measure μ is monogenic if and only if it is singular with respect to every antimonogenic measure.*

PROOF. We prove sufficiency. Suppose μ is singular with respect to every antimonogenic measure. We have $\mu=\mu_0+\mu_1$, where μ_0 is monogenic and μ_1 is antimonogenic. Since $\mu\perp\mu_1$, we have $\mu=\mu_0$.

In view of Theorem 3.5, we see that Theorem 2.1 holds if we take $\mathfrak{N}$ to be the class of monogenic measures.

4. **Examples.** Let I be any uncountable set, and let $X=2^I$ be the space of all functions from I into the discrete space of two elements, with the product topology. If $x\in X$, then x can be written as χ_B, the characteristic function of some $B\subset I$. Now if $x_0\in X$ and μ is the Borel measure on X such that $\mu(E)=1$ or 0 as $x_0\in E$ or $x_0\notin E$, then μ is antimonogenic. Indeed, there exist uncountably many mutually singular Borel measures ν_A such that $\nu_A(\{x_0\})=0$ and such that $\nu_A(U)=1$ if U is an open set containing x_0. Such measures are easily seen to be antiregular relatives of μ.

We show the existence of such measures ν_A for the case $x_0=0$. (An obvious translation handles the general case.) Suppose A is an uncountable subset of I. Let

$$S_A = \{\chi_B \in X : B \subset A \quad \text{and} \quad A - B \text{ is countable}\}.$$

If E is a Borel set, let $\nu_A(E)=1$ if $E\cap S_A$ contains a set F such that F is closed with respect to the relative topology on S_A, and such that given $\chi_B\in S_A$, there exists $\chi_C\in F$ for which $C\subset B$. Otherwise let $\nu_A(E)=0$. Using reasoning similar to that of [2, Exercise 52.10], it can be seen that ν_A has the properties described above. Incidentally, $\nu_{A(1)}$ and $\nu_{A(2)}$ are *distinct* if and only if $A(1)\triangle A(2)$ is uncountable; and when this happens $\nu_{A(1)}$ and $\nu_{A(2)}$ are in fact singular.

Does there exist an antimonogenic Borel measure (or equivalently, an antiregular one) such that each open set has positive measure? The answer is yes, as we now show. Let $I=[0, 1]$, and let $X=2^I$. Such an X is separable, and we let $\{x_n\}$ be a countable dense subset. Define $\mu_n(E)=1/2^n$ or 0 according to whether $x_n \in E$ or $x_n \notin E$. If $\mu = \sum \mu_n$, then μ is antimonogenic since each μ_n is antimonogenic. Of course, each open set has positive measure since $\{x_n\}$ is dense.

References

1. S. K. Berberian, *Measure and integration*, Macmillan, New York, 1965.

2. P. R. Halmos, *Measure theory*, Van Nostrand, New York, 1950.

3. R. A. Johnson, *On the Lebesgue decomposition theorem*, Proc. Amer. Math. Soc. **18** (1967), 628–632.

Washington State University