RATIONAL COHOMOLOGY OPERATIONS AND MASSEY PRODUCTS

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Let Q be the group of rational numbers. Then $H^*(Q, n; Q)$ is either an exterior algebra or polynomial algebra on a class u of dimension n. By the Künneth formula, if $P = \times_{j=1}^{s} K(Q, n_j)$, that is if P is a rational generalized Eilenberg-MacLane space (GEM), then every class in $H^k(P; Q)$ is a polynomial on the fundamental classes $\{u_j\}$. Thus every rational primary cohomology operation on $(x_1, \cdots, x_s)$ can be written $\Phi\{x_j\} = \sum \lambda_j x_j + \sum y_r z_r$ when $\lambda_j \in Q$ and $y_r, z_r \in \tilde{H}^*(P; Q)$. The decomposable term is the twofold matrix Massey product

$$\left\langle (y_1 \cdots y_n) \begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{pmatrix} \right\rangle .$$

In this paper we show that rational higher order cohomology operations can be expressed as a linear term plus a sum of matrix Massey products. As a corollary we conclude that the only stable rational cohomology operations are addition and scalar multiplication.

In defining a rational cohomology operation we recall the notion of a rational Postnikov tower. Let $P_0 = \times_{j=1}^{s} K(Q, n_j)$ where $n_1, \cdots, n_s$ are not necessarily distinct positive integers. We say that

$$\mathcal{P} = \pi \quad \begin{array}{l} P_m \\ \downarrow \pi_m \\ P_{m-1} \xrightarrow{k_m} K(Q, j_m) \\ \downarrow \\ \cdot \\ \cdot \\ \cdot \\ \downarrow \pi_1 \\ P_0 \xrightarrow{k_1} K(Q, j_1) \end{array}$$

is an $m+1$ stage rational Postnikov tower if $1 < j_1 \leq \cdots \leq j_m$ and $P_r \xrightarrow{\pi_r} P_{r-1}$ is the fibration induced from the path loop fibration over $K(Q, j_r)$ by the map k_r.

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For $\mathcal{P}$ a Postnikov tower as above, set $u_j=\pi^*u_j'$, $j=1, \cdots, s$, where $u_j'\in H^{n_j}(P_0; \boldsymbol{Q})$ is the jth fundamental class of P_0, and let $v\in H^k(P_m; \boldsymbol{Q})$ where $k\geqq j_m$. Then the triple $(\mathcal{P}, \{u_j\}, v)$ is the universal example for a higher order cohomology operation Φ defined as follows. For a CW complex X and classes $x_j\in H^{n_j}(X; \boldsymbol{Q})$, $\Phi\{x_j\}$ is defined and contains $y\in H^k(X; \boldsymbol{Q})$ if and only if there is a map $g:X\to P_m$ such that $g^*u_j=x_j$ for $j=1, \cdots, s$ and $g^*v=y$. If Y is an arbitrary space, let $f:X\to Y$ be a weak homotopy equivalence from a CW complex X to Y. Then we set $\Phi\{y_j\}=f^*\Phi\{f^{*-1}y_j\}$ (see [**1**, p. 54–55]).

Definition 1. We say that Φ is a 1-connected rational cohomology operation (of s variables and of degree k) if Φ has a rational universal example $(\mathcal{P}, \{u_j\}, v)$ as described above where dim $u_j>1$ for $j=1, \cdots, s$, and dim $v=k$.

Lemma 2. *Let P be a simply connected space whose rational cohomology has finite type and such that ΩP has the homotopy type of a rational GEM. Let σ be the loop suspension homomorphism. Then every class in* Im σ *is a linear combination of the fundamental classes of ΩP.*

Proof. $H^*(\Omega P; \boldsymbol{Q})$ is a commutative, associative Hopf algebra over $\boldsymbol{Q}$. By Lemma 4.17 of [**5**], the natural map from primitives to indecomposables, $PH^*(\Omega P, \boldsymbol{Q})\to QH^*(\Omega P; \boldsymbol{Q})$, is a monomorphism. As noted in the first paragraph, every class in $H^*(\Omega P; \boldsymbol{Q})$ is a polynomial on the fundamental classes of ΩP. The lemma now follows since Im $\sigma\subset PH^*(\Omega P; \boldsymbol{Q})$.

Lemma 3. *Let $(\mathcal{P}, \{u_j\}, v)$ be the universal example for a* 1*-connected rational cohomology operation. Then ΩP_m has the homotopy type of a rational GEM.*

Proof. Since P_0 is a rational GEM, so is ΩP_0. Assume, inductively, that $\Omega P_n\simeq L\times K(\boldsymbol{Q}^t, q)$ where $\boldsymbol{Q}^t$ is t-dimensional rational vector space, $q=j_{n+1}-1$, and L is a rational GEM with no factor of degree q. ΩP_{n+1} is the fiber space induced by the map $\Omega k:\Omega P_n\to K(\boldsymbol{Q}, q)$. Let z be the fundamental class of $K(\boldsymbol{Q}, q)$. By Lemma 2, since $(\Omega k)^*(z)\in$ Im σ, there is a map $g:K(\boldsymbol{Q}^t, q)\to K(\boldsymbol{Q}, q)$ such that $(\Omega k)^*(z)=p^*g^*(z)$, and so $\Omega k\simeq gp$, where $p:\Omega P_n\to K(\boldsymbol{Q}^t, q)$ is the projection. Thus if E is the fiber space induced by g, then $\Omega P_{n+1}\simeq L\times E$. It remains to show that E has the homotopy type of a rational GEM.

Clearly the homomorphism $g_*:\pi_q(K(\boldsymbol{Q}^t, q))\to\pi_q(K(\boldsymbol{Q}, q))$ is either 0 or an epimorphism. In the first case g itself is null homotopic, so $E\simeq K(\boldsymbol{Q}^t, q)\times K(\boldsymbol{Q}, q-1)$. In the second case the homotopy long exact sequence for the fibration $K(\boldsymbol{Q}, q-1)\to E\to K(\boldsymbol{Q}^t, q)$ implies that $\pi_j(E)=0$ if $j\neq q$ and $\pi_q(E)=\mathbf{Q}^{t-1}$ and so $E=K(\mathbf{Q}^{t-1}, q)$.

THEOREM 4. *Let Φ be a 1-connected rational cohomology operation defined on $\{x_j\}\in H^*(X; \mathbf{Q})$. Then $\Phi\ \{x_j\} = \sum\lambda_j x_j+\mathfrak{U}$, where $\lambda_j\in\mathbf{Q}$ and $\mathfrak{U}$ is a sum of matrix Massey products.*

PROOF. Let $(\mathcal{P}, \{u_j\}, v)$ be the universal example for Φ. Then by Lemma 3, ΩP_m has the weak homotopy type of a rational GEM. By Lemma 2, σv is a linear combination of fundamental classes. Since $j_m\leq k$, these fundamental classes must come from ΩP_0. Thus $\sigma v = \sum\lambda_j\sigma u_j$.

J. P. May (Corollary 18 [4]) has shown that the kernel of σ is generated by matrix Massey products. Since $v-\sum\lambda_j u_j\in\operatorname{Ker}\sigma$, the theorem follows by naturality.

Note that the entries of the matrices in a matrix Massey product are not assumed to be taken from among the fundamental classes. For example we could define a nontrivial Massey triple product of the form $\langle\langle u_1, u_2, u_3\rangle, u_4, u_5\rangle$.

COROLLARY 5. *Let θ be a stable rational cohomology operation (see [1, p. 64]). If θ is defined on $\{x_j\}$ in $\tilde{H}^*(X; \mathbf{Q})$ where X is a connected space, then we can write $\theta\{x_j\} = \sum\lambda_j x_j$ for some $\lambda_j\in\mathbf{Q}$.*

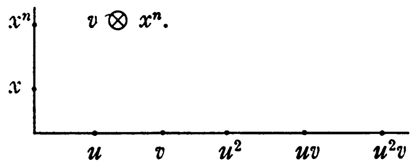
PROOF. Since θ is stable, there is a 1-connected rational cohomology operation Φ such that $s\theta\{x_j\}=\Phi\{sx_j\}$ where $s:H^n(X; \mathbf{Q})\to H^{n+1}(SX; \mathbf{Q})$ is the suspension isomorphism. By Theorem 4, we can write $\Phi\{sx_j\} = \sum\lambda_j sx_j+\mathfrak{U}$. But $\mathfrak{U}$ is a sum of matrix Massey products defined in $H^*(SX; \mathbf{Q})$ and therefore, by the dual of Theorem 5[3], it is identically 0. Thus $\theta\{x_j\} = \sum\lambda_j x_j$.

EXAMPLE 6. Donald W. Kahn [2] has defined a class of secondary cohomology operations with real coefficients which he calls the generalized double and triple products. We shall describe the analogue of these operations in rational cohomology. Let $u\in H^p(X; \mathbf{Q})$ and $v\in H^q(X; \mathbf{Q})$ where p is even, q is odd and $uv=0$. Note that $v^2=0$. Then the rational generalized double product $\langle u, v\rangle_n$ $(n\geqq 1)$ is defined and has dimension $n(p+q)+q-n$.

The universal example for this operation is $(P, \{u, v\}, w_n)$ where P is induced from the "cup product" pairing k

$$\begin{array}{ccc} K(\mathbf{Q}, p+q-1) \xrightarrow{i} P & & \\ \downarrow \pi & & \\ K(\mathbf{Q}, p)\times K(\mathbf{Q}, q) \xrightarrow{k} K(\mathbf{Q}, p+q). & & \end{array}$$

To define w_n we examine the Serre spectral sequence of the above fibration.



In the above diagram, x is the fundamental class of $K(\mathbf{Q}, p+q-1)$ and $d_{p+q}x = uv$. For dimension reasons it is clear that $v \otimes x^n$ in E_2 survives to E_∞. We call w_n the class it represents in $H^{n(p+q)+q-n}(P; \mathbf{Q})$.

It can be shown that $\langle u, v\rangle_n \subset \pm n! < v, \cdots, v, u^n > (n+1)v$'s. For example let a and b be cocycle representatives of u and v respectively and let $\delta c = ab$. Then setting

$$a_{1,1} = a_{2,2} = a, \quad a_{3,3} = b, \quad a_{1,2} = \tfrac{1}{2}(a \cup_1 a), \quad a_{2,3} = -c$$

we have a defining system [3] for $\langle v, v, u\rangle$ with related cocycle $ac - \frac{1}{2}(a \cup_1 a)b$. The class of this cocyle in E_2 is clearly $v \otimes x$.

Thus these secondary operations are actually degenerate higher order Massey products. The generalized triple products can be described in a similar manner.

Bibliography

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