EXTENDING HOMEOMORPHISMS OF $S^p \times S^q$

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The purpose of this note is to characterize those homeomorphisms of $S^p \times S^q$ onto itself which are concordant to the identity, and then to use this to classify, up to concordance, those which extend to homeomorphisms of S^{p+q+1}, regarded as $\partial(B^{p+1} \times B^{q+1}) = S^p \times B^{q+1} \cup B^{p+1} \times S^q$. We work throughout in the PL category, as defined and explicated in either [2] or [4].

Recall that homeomorphisms f_0, f_1 of $S^p \times S^q$ onto itself are said to be *concordant* (equivalently, *weakly isotopic* [3]) if there is a homeomorphism F of $S^p \times S^q \times I$ onto itself such that $F(x, i) = (f_i(x), i)$ for $i = 0, 1$. $I = [0, 1]$ is the unit interval and we shall denote the identity function by 1. We will say that a homeomorphism of $S^p \times S^q$ onto itself is *nice* if it preserves orientation, extends to a homeomorphism of S^{p+q+1}, and induces the identity on $\pi_p(S^p \times S^q)$, where $1 \leqq p \leqq q$.

THEOREM 1. *A homeomorphism h of $S^p \times S^q$ onto itself is nice if and only if it is concordant to the identity.*

PROOF. The case $p = q = 1$ follows from classical results, so we assume $q \geqq p \geqq 1$ and $q + p \geqq 3$. Clearly homeomorphisms concordant to the identity are nice.

Suppose h is a nice homeomorphism of $S^p \times S^q$, and let h': $S^{p+q+1} \to S^{p+q+1}$ be an extension of h; then h' maps each complementary domain onto itself. For $p < q$ this is obvious, and for $p = q$ it follows from the assumption that h induces the identity on $\pi_p(S^p \times S^q)$. Thus h' must also preserve orientation. Restricting h' to $S^p \times B^{q+1}$, it must induce the identity on $\pi_p(S^p \times B^{q+1}) = Z$; hence $h' | S^p \times 0$ is homotopic in $S^p \times B^{q+1}$ to the identity. Applying Zeeman's unknotting Theorem (Chapter 8 of [4], or Volume II of [2]), there is an isotopy F: $S^p \times B^{q+1} \times I \to S^p \times B^{q+1} \times I$ such that $F_0 = 1$, $F | S^p \times S^q \times I = 1$, and $F_1 h' | S^p \times 0 = 1$. Define h'': $S^{p+q+1} \to S^{p+q+1}$ by $h'' = F_1 h' \cup h'$; that is, $h'' | S^p \times B^{q+1} = F_1 h'$ and $h'' | B^{p+1} \times S^q = h'$. Then h'' is an extension of h which is isotopic to h' by an isotopy preserving complementary domain, and such that h'' is the identity on the unknotted sphere $S^p \times 0$. According to Lemma 59, Chapter 8 of [4], there is an isotopy G: $S^{p+q+1} \times I \to S^{p+q+1} \times I$ from h'' to the identity such that $G | S^p \times 0 \times I = 1$. Now both $S^p \times B^{q+1} \times I$ and $G(S^p \times B^{q+1} \times I)$ are regular neighborhoods of $S^p \times 0 \times I$ in $S^{p+q+1} \times I$ meeting the boundary regu-

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larly in the same set; by the uniqueness of regular neighborhoods [1], there is a homeomorphism H of $S^{p+q+1}\times I$ onto itself which is the identity on $S^{p+q+1}\times\partial I$ and such that $HG(S^p\times B^{q+1}\times I)=S^p\times B^{q+1}\times I$. Thus by restricting HG to $S^p\times S^q\times I$ we obtain a concordance from h to the identity, completing the proof of Theorem 1.

Notice that in the above proof HGF is a concordance from h' to the identity which preserves complementary domains. We exploit this to prove the following, which may be considered as a generalization of Theorem 3 of [3].

COROLLARY. *If h is a homeomorphism of $S^p\times B^{q+1}$ onto itself whose restriction to the boundary $S^p\times S^q$ is concordant to the identity, then h is concordant to the identity.*

We do not assume in this corollary that $p\leqq q$. The proof is as follows: use a collar of $S^p\times S^q$ in $B^{p+1}\times S^q$ and the concordance from $h|S^p\times S^q$ to the identity to extend h to all of S^{p+q+1}. Then carry out the proof of Theorem 1; then, as remarked above, HGF restricts to a concordance on $S^p\times B^{q+1}$ from h to the identity.

The set of concordance classes of homeomorphisms of $S^p\times S^q$ onto itself forms a group under composition which we denote by $\mathrm{Conc}(S^p\times S^q)$. Let $G_{p,q}$ be the subgroup consisting of concordance classes of homeomorphisms which extend to S^{p+q+1}. Assume $p\geqq 1$.

THEOREM 2. *$G_{p,q}$ is isomorphic to Z_2+Z_2 if $p<q$, and $G_{p,p}$ is isomorphic to the dihedral group.*

PROOF. Let $f\colon S^p\to S^p$, $g\colon S^q\to S^q$ be orientation reversals, and denote their conical extensions by $C(f)\colon B^{p+1}\to B^{p+1}$, $C(g)\colon B^{q+1}\to B^{q+1}$. Define α, $\beta\colon S^p\times S^q\to S^p\times S^q$ by $\alpha=f\times g$ and $\beta=f\times 1$. We may define extensions of α, β to S^{p+q+1} by $\alpha'=f\times C(g)\cup C(f)\times g$ and $\beta'=f\times 1\cup C(f)\times 1$. By examining their effect on orientation, or on $\pi_p(S^p\times S^q)$, one may see that no two of the homeomorphisms 1, α, β, $\alpha\beta=\beta\alpha$ are concordant. However, $\alpha\alpha$ and $\beta\beta$ are easily seen to be nice and hence concordant to the identity. This shows that the subgroup of $\mathrm{Conc}(S^p\times S^q)$ generated by α and β is isomorphic to Z_2+Z_2; but this subgroup is just $G_{p,q}$ if $p<q$, since one easily verifies that any extendable homeomorphism is concordant to either 1, α, β, or $\alpha\beta$ by noting that its composition with one of them must be nice, and hence concordant to the identity.

If $p=q$, we add another generator λ defined by $\lambda(x, y)=(y, x)$. Clearly λ is extendable, $\lambda\lambda=1$, and λ commutes with α; however, $\lambda\beta$ and $\beta\lambda$ are not concordant since they induce different isomorphisms of $\pi_p(S^p \times S^p)$. It is easy to check that α, β, λ generate a subgroup of $\mathrm{Conc}(S^p \times S^q)$ of order 8; since there are only five groups of order 8, it is easy to determine this group: it is not commutative, and has too many elements of order two to be the Quaternion group, so it must be the dihedral group. But α, β, λ generate $G_{p,p}$; to see this, note that $1, \alpha, \beta, \alpha\beta, \lambda, \lambda\alpha, \lambda\beta, \beta\lambda$ are all distinct, because they induce different automorphisms of $\pi_p(S^p \times S^p)$, and any extendable homeomorphism is concordant to one of them because its composition with one of them must be nice.

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