ON THE DUAL OF HORNICH'S SPACE

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Let D be unit disk and $\mathcal{H}$ the set of analytic functions f in D with $f(0)=0, f'(z)\neq 0$ and $|\arg f'(z)| \leqq K=K_f$, $f'(0)=1$. Hornich has defined operations on $\mathcal{H}$ so that it is a real Banach space [2]. That is if f, g are in $\mathcal{H}$ and α is real

$$[f, g](z) = \int_0^z f'(t)g'(t)dt,$$

$$[\alpha \times f](z) = \int_0^z [f'(t)]^\alpha dt,$$

and $\|f\| = \sup_{z',z''\in D} |\arg f'(z') - \arg f'(z'')|$. Let ν be a real valued, bounded, finitely additive set function on the Lebesgue measurable subsets of $|z|=1$ with $\nu(E)=0$ if E is a Lebsegue null set. We will call ν a charge.

THEOREM. *The dual of the Hornich space on D can be identified isomorphically with equivalence classes of charges on the unit circle.*

PROOF. Let $\mathcal{U}_0$ be the linear space of bounded harmonic functions on D vanishing at the origin. We define a map T from $\mathcal{H}$ to $\mathcal{U}_0$ by $T(f) = \arg f'$. The normalization of functions in $\mathcal{H}$ and the Cauchy-Riemann equations guarantee that T is one to one and onto $\mathcal{U}_0$. Moreover, if $\|u\|_\infty$ is the sup norm for a function $u \in \mathcal{U}_0$ we see that

$$\|u_f\|_\infty \leqq \|f\|_{\mathcal{H}} \leqq 2\|u_f\|_\infty$$

where u_f is the image of f under T. Thus T is a bicontinuous map of $\mathcal{H}$ onto $\mathcal{U}_0$. The Fatou theorem implies that $\lim_{r\to 1} u(re^{i\theta}) = u(\theta)$ exists a.e. Also it can be shown that the essential sup norm of $u(\theta)$ coincides with the sup norm of u in the disk. Further, if $\bar{u}$ is the equivalence class of $u(\theta)$ in L^∞ then it is uniquely determined by u. The dual of L^∞ is given by the charges ν on $|z|=1$, see N. Dunford and J. Schwartz [1, pp. 296–297]. Since $\mathcal{U}_0$ (regarded as a subspace of L^∞) has codimension one we have the result. That is a functional on $\mathcal{H}$ is given by an equivalence class $[\nu]$ of charges, where $\nu_1 \sim \nu_2$ if and only if $\nu_1 = \nu_2 + k d\theta$, k a constant.

We observe that the obvious linear functionals

Received by the editors September 11, 1968 and, in revised form, October 28, 1968.

$$T_{z_0}(f) = \arg f'(z_0), \qquad |z_0| < 1$$

are given by integrating the boundary values of arg f' against the Poisson kernel at z_0. Also one can show by examples that the evaluation maps $f \to f(z_0)$ are not continuous in the Hornich topology.

REFERENCES

1. N. Dunford and J. Schwartz, *Linear operators*. Part I, Interscience, New York, 1958.

2. H. Hornich, *Ein Banachraum analytischer Funktionen in Zusammenhang mit den schlichten Funktionen*, Monatsh. Math. **73** (1969), 36–45.

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